

1. Consider the following matrices and perform the operations listed when possible:

$$A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & 3 & 1 \\ 2 & -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

a. Find the rank and determinant of each of the above matrices.

b. Calculate AB , $(AB)^T C$, BA , $2A - B$, and $3A + C$.

c. Find the eigenvalues and a basis of eigenvectors for A and C . Determine the algebraic and geometric multiplicity of each of the eigenvalues.

2. Consider the following matrix:

$$Q = \begin{pmatrix} 4 + a & -2 - a & 0 & -4 \\ 4 & -2 & 0 & -4 \\ 0 & 3 - a & 3 & -3 + a \\ 4 & -2 - a & 0 & -4 + a \end{pmatrix}.$$

a. Find the rank and determinant of Q . Find the eigenvalues and eigenvectors of Q . Be sure to give the algebraic and geometric multiplicity of the eigenvalues (noting all special cases for particular values of a).

b. Consider the system of equations

$$Qx = 0.$$

Find what values of a give non-trivial solutions and find those solutions. What is the solution for $a = 5$?

c. Diagonalize Q , if possible. What values of a result in Q being non-diagonalizable and why?

3. Solve the following system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 4, \\ 2x_1 + x_2 + 3x_5 &= 2, \\ -2x_1 + 2x_3 + x_4 - 4x_5 &= 0, \\ x_2 + 2x_3 + x_4 - x_5 &= 2. \end{aligned}$$

4. Consider the following matrix:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & -c \\ b & c & 0 \end{pmatrix}.$$

a. Find the determinant, eigenvalues, and eigenvectors for this matrix.

b. Does the set of matrices of the type above (with a , b , and c arbitrary) form a vector space? If so, find a basis and determine its dimension. If not, why?

5. a. For which values of α will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned}x_1 + x_2 &= 3, \\2x_1 - x_2 - 3x_3 &= 0, \\6x_1 - 3x_2 - \alpha^2x_3 &= \alpha + 3.\end{aligned}$$

b. Solve this system for $\alpha = -3$.

6. a. Form the vector space V of cubic polynomials without any first order terms

$$V = \{p|p(x) = a_3x^3 + a_2x^2 + a_0, \text{ for } x \in [0, 1], a_i \in \mathbb{R}\}.$$

Find a basis and the dimension of this vector space.

b. Form the subspace W of V with $p(0) = 0$. Find the dimension of this subspace and find a basis.

c. Define an inner product for part b. as follows:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Clearly, $p_3(x) = x^3$ is an element of W . Find an element in W orthogonal to p_3 .

7. The electric circuit presented below has the following resistances (in ohms) and electromotive forces (in volts):

$$R_0 = 15, R_1 = 10, R_2 = 20, R_3 = 5, R_4 = 30, R_5 = 20, E_0 = 50, \text{ and } E_5 = 100.$$

Use Kirchhoff's laws to determine the unknown currents.

