Math 342A Exam 3 Fall 2004

① a. $y' - \frac{2}{x}y = 4x^2\sin(4x)$, $\mu(x) = \exp(\int -\frac{2}{x}dx) = e^{-2\ln(x)} = x^{-2}$, $\frac{d}{dx}(x^{-2}y) = 4\sin(4x)$

$\Rightarrow x^{-2}y = 4\int\sin(4x)dx + c$ ∴ $y(x) = x^2(C - \cos(4x))$

b. $y' - y = y^{-1}xe^{3x}$ (Bernoulli) $u = y^{1-(-1)} = y^2$, $u' = 2yy'$ ∴ $2yy' - 2y^2 = 2xe^{3x}$ or $u' - 2u = 2xe^{3x}$

$\mu(x) = e^{-2x}$. $\frac{d}{dx}(e^{-2x}u) = 2xe^x \Rightarrow e^{-2x}u = 2\int xe^x dx + c = 2(x-1)e^x + c$ ∴ $u = y^2 = 2(x-1)e^{3x} + Ce^{2x}$

② a. $xy' = 1 + y^2$ (separable) $\int\frac{dx}{1+y^2} = \int\frac{dx}{x} \Rightarrow \arctan(y) = \ln|x| + C$, $y(1) = 1 \Rightarrow \arctan(1) = \frac{\pi}{4} = \ln|1| + c$

$c = \pi/4$ ∴ $\arctan(y) = \ln|x| + \pi/4$ or $y(x) = \tan(\ln|x| + \pi/4)$.

b. $y' = -y/2$, $y(x) = 4e^{-x/2}$.

③ a. $y' = -2t(y-2) \Rightarrow y' + 2ty = 4t$, $\mu(t) = e^{t^2}$ ∴ $\frac{d}{dt}(e^{t^2}y) = 4te^{t^2} \Rightarrow e^{t^2}y = 4\int te^{t^2}dt$

0 $e^{t^2}y = 2e^{t^2} + c$ ∴ $y(t) = 2 + ce^{-t^2}$, $y(0) = 6 \Rightarrow c = 4$ Thus, $y(t) = 2 + 4e^{-t^2}$

b. $y_{n+1} = y_n + h(-2t_n(y_n - 2))$ with $h = 0.2$ $y_{n+1} = y_n - 0.4t_n(y_n - 2)$

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t_n	y_n
$t_0 = 0$	$y_0 = 6$
$t_1 = 0.2$	$y_1 = y_0 - 0.4t_0(y_0 - 2) = 6$
$t_2 = 0.4$	$y_2 = y_1 - 0.4t_1(y_1 - 2) = 6 - 0.08(4) = 5.68$
$t_3 = 0.6$	$y_3 = 5.68 - 0.4(0.4)(5.68 - 2) = 5.0412$

$y(.6) = 2 + 4e^{-.6^2} = 2 + 4e^{-.36} = 4.7907$

% Error = $\frac{5.0412 - 4.7907}{4.7907} \times 100 = 6.27\%$

④ a. $\frac{dT_1}{dt} = -k(T_1 - 0)$, $T_1(0) = 50 \Rightarrow T_1(t) = 50e^{-kt}$, $T_1(4) = 25 = 50e^{-4k}$ ∴ $e^{4k} = 2$

$k = \frac{1}{4}\ln(2) \approx 0.173287$. $T_1(t) = 50e^{-kt} = 10 \Rightarrow e^{kt} = 5$, $t = \frac{\ln(5)}{k} \approx 9.288$ hrs.

b. $\frac{dT_2}{dt} + \ln(2)T_2 = 50\ln(2)e^{-kt}$, $\mu(t) = e^{t\ln(2)}$, $\frac{d}{dt}\left(e^{t\ln(2)}T_2(t)\right) = 50\ln(2)e^{(\ln(2) - \frac{1}{4}\ln(2))t}$

8 $e^{t\ln(2)}T_2(t) = \frac{50\ln(2)}{\frac{3}{4}\ln(2)}e^{\frac{3}{4}\ln(2)t} + c \Rightarrow T_2(t) = \frac{200}{3}e^{-\frac{\ln(2)}{4}t} + Ce^{-t\ln(2)}$ $T_2(0) = 0 \Rightarrow c = -\frac{200}{3}$

∴ $T_2(t) = \frac{200}{3}\left(e^{-t\frac{\ln(2)}{4}} - e^{-t\ln(2)}\right)$ Max when $\frac{dT_2}{dt} = 0 \Rightarrow -\frac{\ln(2)}{4}e^{-\frac{t\ln(2)}{4}} + \ln(2)e^{-t\ln(2)} = 0$

$\Rightarrow \frac{1}{4}e^{-\frac{t\ln(2)}{4}} = e^{-t\ln(2)}$ or $e^{\frac{3}{4}t\ln(2)} = 4$ ∴ $t = \frac{4\ln(4)}{3\ln(2)} = \frac{8}{3}$ hr $T_2(8/3) = \frac{200}{3}\left(e^{-\frac{2\ln(2)}{3}} - e^{-\frac{8\ln(2)}{3}}\right)$

$\Rightarrow T_2(8/3) = 5.997°C$ is max temp.

c. $\frac{dT_1}{dt} = -kT_1^4$ ∴ $-\int\frac{dT_1}{T_1^4} = \int k\,dt = kt + C = \frac{1}{3T_1^3} \Rightarrow T_1^3(t) = \frac{1}{3kt + \hat{C}}$, $T_1(t) = \frac{1}{\sqrt[3]{3kt + \hat{C}}}$

12 $T_1(0) = 50 = \frac{1}{\sqrt[3]{\hat{c}}} \Rightarrow \hat{c} = \frac{1}{125000}$, $T_1(4) = 25 = \frac{1}{\sqrt[3]{\hat{c} + 12k}}$ ∴ $12k + \hat{c} = \frac{1}{15625}$,

$k = \frac{1}{12}\left(\frac{1}{15625} - \frac{1}{125000}\right) = \frac{7}{12}\frac{1}{125000} \approx 4.67 \times 10^{-6}$