

① a. rank(A) = 3, rank(B) = 2, rank(C) = 2, det(A) = 2, det(B) undefined, det(C) = 0

$$b. AB = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 0 & -2 \\ 1 & 2 \end{pmatrix}, \quad B^T = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad BA \text{ undefined}, \quad A - 2C = \begin{pmatrix} 3 & -2 & 3 \\ -2 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \left(\begin{array}{c} \leftarrow -2 \left| \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right| \end{array} \right)$$

$$c. \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -(\lambda-1)^2(\lambda-2) = 0, \quad \lambda_1 = 2, \lambda_2 = 1 \quad \left(\text{For } \lambda_1 = 2, \begin{vmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} \xi = 0 \therefore \text{e.f. } \xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ alg. + geo mult} = 1 \right)$$

$$\text{For } \lambda_2 = 1 \quad \begin{vmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \xi = 0 \therefore \text{e.f. } \xi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \text{alg. mult} = 2 \\ \text{geo. mult} = 1 \end{array}$$

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = -(\lambda+2)[(\lambda+1)^2 - 1] = -\lambda(\lambda+2)^2 = 0 \quad \left(\text{For } \lambda_1 = 0, \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \xi = 0 \therefore \text{e.f. } \xi_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ alg. + geo mult} = 1 \right)$$

$$\text{For } \lambda_2 = -2 \quad \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \xi = 0 \Rightarrow \text{e.f. } \xi_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{alg. + geo mult.} = 2.$$

$$\textcircled{2} \begin{bmatrix} 1 & -2 & 1 & 0 & : & 5 \\ 0 & 0 & -2 & 2 & : & -4 \\ -1 & 2 & 2 & -3 & : & 1 \end{bmatrix} \xrightarrow[\substack{-\frac{1}{2}R_2 \\ R_3+R_1}]{\substack{R_2 \times 2 \\ R_3 - R_1}} \begin{bmatrix} 1 & -2 & 1 & 0 & : & 5 \\ 0 & 0 & 1 & -1 & : & 2 \\ 0 & 0 & 3 & -3 & : & 6 \end{bmatrix} \xrightarrow[\substack{R_3 - 3R_2}]{R_1 - R_2} \begin{bmatrix} 1 & -2 & 0 & 1 & : & 3 \\ 0 & 0 & 1 & -1 & : & 2 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} \quad \begin{array}{l} x_2 = \alpha, \quad x_4 = \beta \\ x_1 = 3 + 2\alpha - \beta, \quad x_3 = 2 + \beta \end{array}$$

$$\therefore [x_1, x_2, x_3, x_4]^T = \alpha [2 \ 1 \ 0 \ 0]^T + \beta [-1 \ 0 \ 1 \ 1]^T + [3 \ 0 \ 2 \ 0]^T$$

$$\textcircled{3} \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 2 & 0 & 3 & : & 4 \\ 1 & -1 & \beta^2 & : & \beta+1 \end{bmatrix} \xrightarrow[\substack{R_3 - R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 0 & -2 & 5 & : & 2 \\ 0 & -2 & \beta^2+1 & : & \beta \end{bmatrix} \xrightarrow[\substack{R_3 - R_2}]{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & -1 & : & 1 \\ 0 & 1 & -5/2 & : & -1 \\ 0 & 0 & \beta^2-4 & : & \beta-2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 3/2 & : & 2 \\ 0 & 1 & -5/2 & : & -1 \\ 0 & 0 & \beta^2-4 & : & \beta-2 \end{bmatrix}$$

a. No solution $\beta = -2$, Infinitely many $\beta = 2$, Unique $\beta \neq \pm 2$.

$$b. \text{ If } \beta = -1, \begin{bmatrix} 1 & 0 & 3/2 & : & 2 \\ 0 & 1 & -5/2 & : & -1 \\ 0 & 0 & -3 & : & -3 \end{bmatrix}, \quad x = \begin{pmatrix} 1/2 \\ 3/2 \\ 1 \end{pmatrix}, \quad \text{If } \beta = 2, \begin{bmatrix} 1 & 0 & 3/2 & : & 2 \\ 0 & 1 & -5/2 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}, \quad x = \alpha \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\textcircled{4} a. \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2 \quad b. \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (\lambda-a)^2 - b^2 = 0 \Rightarrow \lambda_1 = a+b, \quad \lambda_2 = a-b.$$

$$\text{For } \lambda_1 = a+b, \quad \begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \xi = 0, \quad \xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \text{For } \lambda_2 = a-b, \quad \begin{pmatrix} b & b \\ b & b \end{pmatrix} \xi = 0, \quad \xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \xi_1 \cdot \xi_2 = 0 \therefore \text{orthogonal}$$

c. This produces a vector space. Basis $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, V = \{x \mid x = \alpha e_1 + \beta e_2\} \dim(V) = 2$

⑤ a. $V = C^1[-1, 1]$ is not a vector space. Suppose $y_1 \in V$ & $y_2 \in V$, then $(y_1 + y_2)' - 2(y_1 + y_2) = (y_1' - 2y_1) + (y_2' - 2y_2) = 1 + 1 = 2$, so $y_1 + y_2 \notin V$.

b. Let α, β be scalars and $y_1, y_2 \in W$. Must show $\alpha y_1 + \beta y_2 \in W$. But $(\alpha y_1 + \beta y_2)'' - (\alpha y_1 + \beta y_2) = \alpha(y_1'' - y_1) + \beta(y_2'' - y_2) = 0 \Rightarrow \alpha y_1 + \beta y_2 \in W$

c. $y_1(x) = e^x, \quad y_1'' - y_1 = e^x - e^x = 0, \quad y_2(x) = e^{-x}, \quad y_2'' - y_2 = e^{-x} - e^{-x} = 0 \therefore y_1, y_2 \in W$

$$\langle y_1, \alpha y_1 + \beta y_2 \rangle = \int_{-1}^1 (\alpha e^{2x} + \beta) dx = \frac{\alpha}{2} e^{2x} \Big|_{-1}^1 + \beta x \Big|_{-1}^1 = \frac{\alpha}{2} (e^2 - e^{-2}) + 2\beta = 0 \Rightarrow \alpha \sinh(2) = -2\beta$$

$$\therefore \text{Take } \beta = -\frac{\alpha \sinh(2)}{2} \therefore \text{orthogonal fcn to } y_1(x) \text{ is } y(x) = \alpha \left(e^x - \frac{e^{-x} \sinh(2)}{2} \right)$$