Math 342A — Exam 2 — Fall 2004

① a. $\text{rank}(A)=3$, $\text{rank}(B)=2$, $\text{rank}(C)=2$, $\det(A)=2$, $\det(B)$ undefined, $\det(C)=0$ $\leftarrow -2\begin{vmatrix}-1&1\\1&-1\end{vmatrix}$

b $AB=\begin{pmatrix}1&0&3\\0&2&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&1\\0&-1\\1&2\end{pmatrix}=\begin{pmatrix}2&7\\0&-2\\1&2\end{pmatrix}$, $B^T=\begin{pmatrix}-1&0&1\\1&-1&2\end{pmatrix}$, BA undefined, $A-2C=\begin{pmatrix}3&-2&3\\-2&4&0\\0&0&5\end{pmatrix}$

c. $\begin{vmatrix}1-\lambda&0&3\\0&2-\lambda&0\\0&0&1-\lambda\end{vmatrix}=-(\lambda-1)^2(\lambda-2)=0$, $\lambda_1=2,\ \lambda_2=1$ For $\lambda_1=2$, $\begin{pmatrix}-1&0&3\\0&0&0\\0&0&-1\end{pmatrix}\xi=0 \therefore$ e.f. $\xi_1=\begin{pmatrix}0\\1\\0\end{pmatrix}$ alg. + geo mult = 1

For $\lambda_2=1$ $\begin{pmatrix}0&0&3\\0&1&0\\0&0&0\end{pmatrix}\xi=0 \therefore$ e.f. $\xi_2=\begin{pmatrix}1\\0\\0\end{pmatrix}$ alg. mult = 2, geo. mult = 1

$\begin{vmatrix}-1-\lambda&1&0\\1&-1-\lambda&0\\0&0&-2-\lambda\end{vmatrix}=-(\lambda+2)[(\lambda+1)^2-1]=-\lambda(\lambda+2)^2=0$, $\lambda_1=0$, $\lambda_2=-2$ For $\lambda_1=0$, $\begin{pmatrix}-1&1&0\\1&-1&0\\0&0&-2\end{pmatrix}\xi=0 \therefore$ e.f. $\xi_1=\begin{pmatrix}1\\1\\0\end{pmatrix}$ alg. + geo mult = 1

For $\lambda_2=-2$ $\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}\xi=0 \Rightarrow$ e.f. $\xi_2=\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\xi_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$ alg. + geo mult. = 2.

② $\left[\begin{array}{cccc|c}1&-2&1&0&5\\0&0&-2&2&-4\\-1&2&2&-3&1\end{array}\right]\xrightarrow[R_3+R_1]{-\frac{1}{2}R_2}\left[\begin{array}{cccc|c}1&-2&1&0&5\\0&0&1&-1&2\\0&0&3&-3&6\end{array}\right]\xrightarrow[R_3-3R_2]{R_1-R_2}\left[\begin{array}{cccc|c}1&-2&0&1&3\\0&0&1&-1&2\\0&0&0&0&0\end{array}\right]$ $x_2=\alpha$, $x_4=\beta$, $x_1=3+2\alpha-\beta$, $x_3=2+\beta$

$\therefore [x_1,x_2,x_3,x_4]^T=\alpha[2\ 1\ 0\ 0]^T+\beta[-1\ 0\ 1\ 1]^T+[3\ 0\ 2\ 0]^T$

③ $\left[\begin{array}{ccc|c}1&1&-1&1\\2&0&3&4\\1&-1&\beta^2&\beta+1\end{array}\right]\xrightarrow[R_3-R_1]{R_2-2R_1}\left[\begin{array}{ccc|c}1&1&-1&1\\0&-2&5&2\\0&-2&\beta^2+1&\beta\end{array}\right]\xrightarrow[R_3-R_2]{-\frac{1}{2}R_2}\left[\begin{array}{ccc|c}1&1&-1&1\\0&1&-5/2&-1\\0&0&\beta^2-4&\beta-2\end{array}\right]\xrightarrow{R_1-R_2}\left[\begin{array}{ccc|c}1&0&3/2&2\\0&1&-5/2&-1\\0&0&\beta^2-4&\beta-2\end{array}\right]$

a. No Solution $\beta=-2$, Infinitely many $\beta=2$, Unique $\beta\neq\pm 2$.

b. If $\beta=-1$, $\left[\begin{array}{ccc|c}1&0&3/2&2\\0&1&-5/2&-1\\0&0&-3&-3\end{array}\right]$, $x=\begin{pmatrix}1/2\\3/2\\1\end{pmatrix}$, If $\beta=2$, $\left[\begin{array}{ccc|c}1&0&3/2&2\\0&1&-5/2&-1\\0&0&0&0\end{array}\right]$ $x=\alpha\begin{pmatrix}-3\\5\\2\end{pmatrix}+\begin{pmatrix}2\\-1\\0\end{pmatrix}$

④ a. $\begin{vmatrix}a&b\\b&a\end{vmatrix}=a^2-b^2$ b. $\begin{vmatrix}a-\lambda&b\\b&a-\lambda\end{vmatrix}=(\lambda-a)^2-b^2=0 \Rightarrow \lambda_1=a+b,\ \lambda_2=a-b$.

For $\lambda_1=a+b$, $\begin{pmatrix}-b&b\\b&-b\end{pmatrix}\xi=0$, $\xi_1=\begin{pmatrix}1\\1\end{pmatrix}$. For $\lambda_2=a-b$, $\begin{pmatrix}b&b\\b&b\end{pmatrix}\xi=0$, $\xi_2=\begin{pmatrix}1\\-1\end{pmatrix}$ $\xi_1\cdot\xi_2=0$ $\therefore$ orthogonal

c. This produces a vector space. Basis $e_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $e_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $V=\{x\mid x=ae_1+be_2\}$ $\dim(V)=2$

⑤ a. $V=C^1[-1,1]$ is not a vector space. Suppose $y_1\in V$ + $y_2\in V$, then $(y_1+y_2)'-2(y_1+y_2)=$
7 $(y_1'-2y_1)+(y_2'-2y_2)=1+1=2$, so $y_1+y_2\notin V$.

b. Let α,β be scalars and y_1 + $y_2\in W$. Must show $\alpha y_1+\beta y_2\in W$. But $(\alpha y_1+\beta y_2)''-(\alpha y_1+\beta y_2)$
7 $=\alpha(y_1''-y_1)+\beta(y_2''-y_2)=0 \Rightarrow \alpha y_1+\beta y_2\in W$

c. $y_1(x)=e^x$, $y_1''-y_1=e^x-e^x=0$, $y_2(t)=e^{-t}$, $y_2''-y_2=e^{-t}-e^{-t}=0$ $\therefore y_1,y_2\in W$

7 $\langle y_1,\alpha y_1+\beta y_2\rangle=\int_{-1}^{1}(\alpha e^{2t}+\beta)\,dt=\frac{\alpha}{2}e^{2t}\Big|_{-1}^{1}+\beta t\Big|_{-1}^{1}=\frac{\alpha}{2}(e^2-e^{-2})+2\beta=0 \Rightarrow \alpha\sinh(2)=-2\beta$

$\therefore$ Take $\beta=-\frac{\alpha\sinh(2)}{2}$ $\therefore$ orthogonal fcn to $y_1(t)$ is $y(t)=\alpha\left(e^t-\frac{e^{-t}\sinh(2)}{2}\right)$