Some Tips on Using a Numerical Solver

A numerical ODE solver plots an approximate value of a solution of an IVP at hundreds of different instants in time and then connects these points on the computer screen with line segments. How well this graph approximates the true solution curve depends on the sophistication of the solver and the character of the ODE being solved. Numerical analysts have done a remarkable job in coming up with reliable solvers. We have a great deal of confidence in our solver, but it pays to use all available information about an ODE before accepting the output of any solver.

For now we are only concerned with the basics of how to communicate with the solver. The first thing to do is to write the IVP in the normal form

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{1}$$

because the numerical solver has to know the *rate function* $f(t, y)$ and the *initial point* (t_0, y_0). Next, the user needs to specify the *solve-time interval* as running from the initial time t_0 to some final time t_1. The IVP is solved *forward* if $t_1 > t_0$, and *backward* if $t_1 < t_0$.

We must tell the solver how to display solution curves. Our solver plots solution values as they are computed; others wait until computation is complete and then scale the axes accordingly. We like to select the axis ranges before using our solver instead of letting the solver select the axis ranges on its own. There are two reasons for this:

- Preselection of axis ranges prevents the computer from working too hard (or crashing). Many solvers shut down automatically when the solution curve gets too far beyond the specified screen area because of a poorly selected solve-time interval.
- Some solvers have a default setting that scales the screen size to the solution curve over the solve-time interval. In that case not much detail will appear on the computer screen when dealing with a runaway solution curve.

☞ Be sure to select axis ranges so that the initial point (t_0, y_0) appears on the screen.

Choosing a screen size to bring out the features you wish to examine is as much an art as it is a science. Your skill at setting screen sizes will improve with experience.

Solvers designed by numerical analysts and coded by experts implement a variety of solver engines, but they may be limited by the platform on which they run. The number of digits of precision used by the solver and accepted as input from the keyboard may vary from one solver/platform to another. These variations usually cause only slight differences in the output over a short time span for the same IVP.

☞ There are more tips on using solvers in the SPOTLIGHT ON APPROXIMATE NUMERICAL SOLUTIONS and in the SPOTLIGHT ON COMPUTER IMPLEMENTATION.

The solver interface has a different look-and-feel from one solver package to another. Interfaces range from those that are largely point-and-click to ones requiring a complex syntax to communicate with the solver. Many solvers can store interactive scripts to bypass the trickier features of the interface. Some archives can be accessed over the Internet. Some solvers (e.g., ODE Architect) contain libraries of interactive scripts.

Commercial solver packages may have features that are set automatically or by default and hence may be invisible to a beginning user. First off, there is the actual solver engine used. A choice may be provided, but the package designer has selected one all-around solver engine to come up by default. Solver engine parameters such as error bounds, step size, and number of points plotted per unit time are usually set adaptively or by default. The user can override some of these defaults.

Solution Curves, Slope Fields, Nullclines

Let's start with the IVP (1), where the rate function $f(t, y)$ and its y-partial derivative $\partial f(t, y)/\partial y$ are continuous functions of t and y in a rectangle R in the ty-plane and (t_0, y_0) is a point inside R. The Existence and Uniqueness Theorem guarantees that IVP (1) has exactly one solution, $y = y(t)$, whose solution curve lies inside R for all values of t in some interval that contains t_0 in its interior. In fact, as we shall see in the next section, under these conditions every solution curve that originates in R must reach the boundary of R as t goes backward and forward from the initial time t_0.

The Existence and Uniqueness Theorem 2.3.1 implies the following very important property that will come in handy when visualizing solution curves.

THEOREM 2.4.1

Solution Curves Don't Intersect

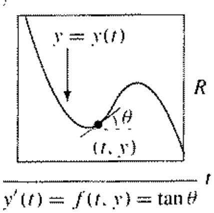
If f and $\partial f/\partial y$ are continuous on a rectangle R, then two different solution curves of the ODE $y' = f(t, y)$ can't intersect in R.

In other words ...lution curves that do ...eet must be arcs of a ...ngle solution curve.

Proof. Say that two distinct solution curves *do* meet at some point (t_0, y_0) in R. Then the IVP $y' = f(t, y)$, $y(t_0) = y_0$, would have two distinct solutions on some interval I containing t_0, contradicting the uniqueness part of Theorem 2.3.1. Thus, the solution curves can't meet after all. ▪

Thus, under the stated conditions on the rate function $f(t, y)$, we can imagine *all* of R to be covered by solution curves that never touch and extend from edge to edge of R.

There is a way to view the solvability of IVP (1) that appeals to geometric intuition and lends itself to a graphical approach to finding solution curves. The ODE says that at each point (t, y) in R the number $f(t, y)$ is the slope of the tangent line to the solution curve through that point (see the margin figure).



$y'(t) = f(t, y) = \tan\theta$

The tangent line at ...oint (t, y) of a solution ...s slope $f(t, y)$.

On the other hand, suppose that the graph of a function $y(t)$ lies in R. Then $y(t)$ defines a solution curve of $y' = f(t, y)$ if at each point (t, y) on its graph the slope of the tangent line has the value $f(t, y)$. This is the geometric way of saying that $y'(t) = f(t, y(t))$; that is, $y(t)$ is a solution for the ODE.

This change of viewpoint gives us an imaginative way to see solution curves for the ODE. By drawing short line segments with slopes $f(t, y)$ and centered at a grid of points (t, y) in R, we obtain a *slope field*. A slope field suggests curves in R with the property that at each point on each curve the tangent line to the curve at that point lies along the line segment of the slope field at the point. This process reveals solution curves in much the same way as sprinkling iron filings on paper and holding it over the poles of a magnet reveals magnetic field lines.

We can draw the line segments of a slope field by hand, but it is much easier to

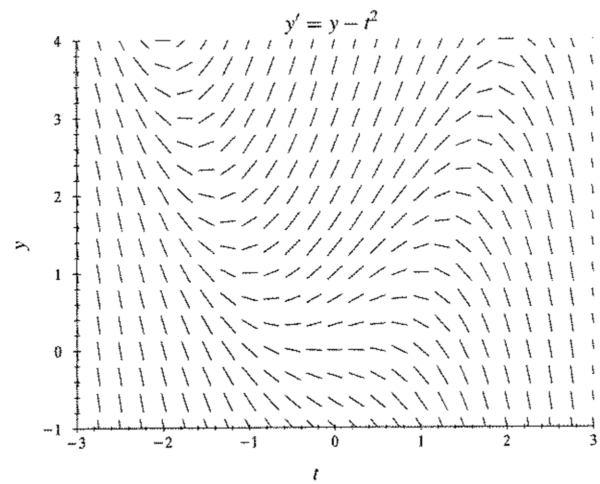


FIGURE 2.4.1 Slope field (Example 2.4.1).

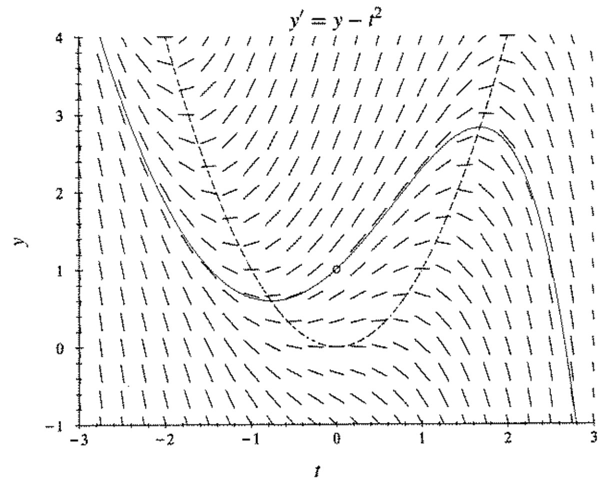


FIGURE 2.4.2 Solution curve (solid) through the point (0, 1); nullcline (dashed) (Example 2.4.1).

have a numerical solver do the work.[4] Most solvers can do this, and many let the user choose the density of the grid points and the length of the field line segments. Before we illustrate these ideas with examples, let's look at curves in the ty-plane that give us some information about the behavior of solution curves.

The *nullclines* (the curves of zero inclination) of $y' = f(t, y)$ are curves defined by $f(t, y) = 0$. Solution curves cross the nullcline with zero slope ($f(t, y) = 0$) because the line segment centered at a point on the nullcline is horizontal. Nullclines are solution curves only for constant solutions. See the WEB SPOTLIGHT ON SLOPE FIELDS I for more on nullclines and slope fields.

A good way to develop confidence in your solver is to use it to reproduce some of the graphs in this text. Keep in mind that these graphs are *not* artist renderings, but actual output from our solver. It also helps to know that data needed to produce these graphs appears in and around the graphs or in the associated text. ODEs with piecewise continuous driving terms can be handled by numerical solvers that accept step functions, pulses, and other engineering functions.

EXAMPLE 2.4.1

☞ Warm up your numerical solver with the IVP $y' = y - t^2$, $y(0) = 1$.

A Slope Field and a Solution Curve

Figure 2.4.1 shows a slope field for the ODE

$$y' = y - t^2$$

You can almost see the solution curves. They rise wherever $y > t^2$ because y' is positive there, and similarly the curves fall where $y < t^2$. The field line segments in the figure

[4]General-purpose mathematical software packages such as Maple, Matlab, and Mathematica have this slope field feature, as do differential equations software packages such as IDE, ODE Architect, and dfield. Some hand-held calculators manufactured by Casio, Sharp, and Texas Instruments also have this capability.

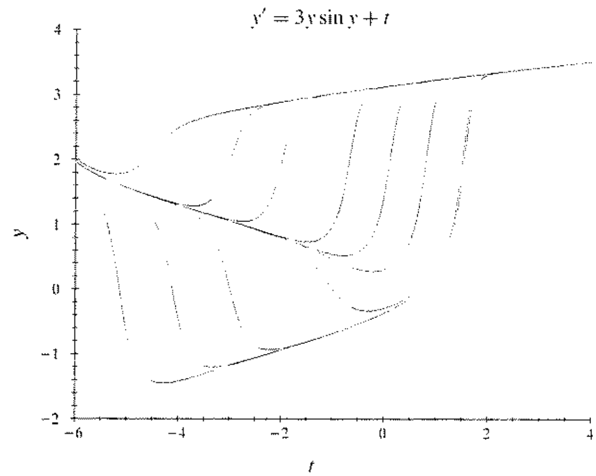


FIGURE 2.4.3 Diverging and then converging solution curves (Example 2.4.2).

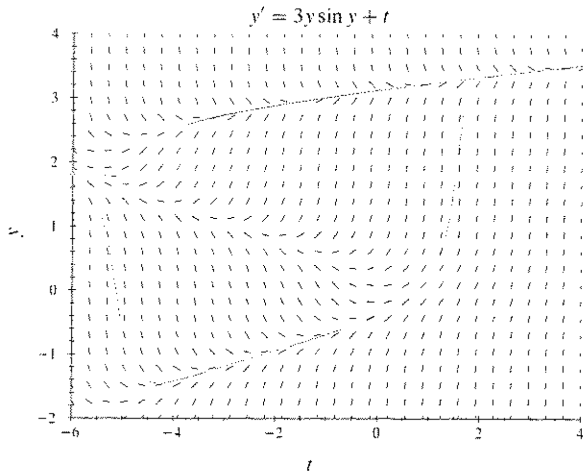


FIGURE 2.4.4 A slope field plot with two solution curves for the ODE in Figure 2.4.3.

☞ The aspect ratio affects the look of curves; a circle may look like an ellipse.

are all the same length, even if they do not appear so at first glance. The reason for this is that the computer screen length of a vertical unit is not the same as the screen length of a horizontal unit. The ratio of the former to the latter is the *aspect ratio* of the display.

In the ODE $y' = y - t^2$, we see that $f(t, y) = y - t^2$, so the nullcline is the parabola defined by $y = t^2$ (the dashed curve in Figure 2.4.2). The nullcline divides the ty-plane into the region above the parabola ($y > t^2$), where solution curves rise ($f(t, y) > 0$) and the region below ($y < t^2$), where they fall ($f(t, y) < 0$).

Now let's use a numerical solver to plot the solution curve through the initial point $t_0 = 0$, $y_0 = 1$. Figure 2.4.2 shows this curve (solid) extended forward and backward in time from $t_0 = 0$ until the curve leaves the rectangle defined by the computer screen. Notice how nicely the solution curve fits the slope field. We used a numerical differential equation solver to approximate and plot the solution curve; we did not use a solution formula, although there is one for this first-order linear ODE.

If the rate function f satisfies the conditions of the Existence and Uniqueness Theorem 2.3.1 in the rectangle defined by your computer screen, then by Theorem 2.4.1, any apparent meeting of solution curves on the screen is only due to the finite resolution capability of the display device. As we shall soon see, zooming helps separate computed solution curves.

Let's illustrate the geometry of solution curves with some examples.

EXAMPLE 2.4.2

☞ See the ODE Architect Library entry *A First-Order ODE with Data Compression* under First Order Equations.

Solution Curves That Seem to Touch

The function $f(t, y) = 3y\sin y + t$ is continuous for all values of t and y, as is $\partial f/\partial y = 3\sin y + 3y\cos y$. So Theorems 2.3.1 and 2.4.1 tell us that through every point (t_0, y_0) of the ty-plane there passes exactly one solution curve of the ODE

$$y' = 3y\sin y + t \tag{2}$$

and that the entire ty-plane is covered with solution curves that never meet.

Figure 2.4.3 shows several solution curves of ODE (2) as plotted by our numerical solver. The two solution curves starting at $t_0 = -6$, $y_0 = 1.8, 2.0$, appear to enclose a large region on the screen. The interior solution curves start close together on the y-axis at $t = -6$ and seem to spread apart as t increases, then eventually flow back together. This makes it nearly impossible to use a numerical solver to generate the specific curves shown inside the region by starting at $t_0 = -6$. (Reason: the initial values y_0 must be known to an accuracy that exceeds the ability of our platform and solver to handle.) To plot these interior curves, we started with initial points *inside* the region and solved forward and backward in time until the solution curves left the screen. The plotted solution curves appear to touch, but that is an illusion caused by finite pixel size. You can't always believe everything you see in a computer display!

☞ See Problem 17 for more on ODE (2).

EXAMPLE 2.4.3

Slope Field for the ODE $y' = 3y\sin y + t$

The slope field in Figure 2.4.4 shows why it is difficult to generate the solution curves shown in Figure 2.4.3 by starting at the initial time $t_0 = -6$. The slope field suggests that all solution curves between the two indicated solution curves are strongly attracted to one or the other of these two solution curves. That's why in Figure 2.4.3 we had to select initial points inside the enclosed region so that our solver could generate the interior solution curves by solving forward and backward. See Problem 17.

Qualitative Information about Solution Behavior

As we saw in Example 2.4.1, a slope field for a first-order differential equation suggests the behavior of solution curves. In that example, we saw that solution curves rise above the parabolic nullcline $y = t^2$, while they fall in the region below the parabola. This is an example of *qualitative* information about solution behavior. Let's look at another example where the slope field guides the eye to see how solution curves behave in a qualitative sense, even though there is little that is quantitative to go on.

EXAMPLE 2.4.4

Time-Dependent Harvesting

Our solver produced the slope field and solution curves in Figure 2.4.5 for a logistic fish population model that is harvested and restocked sinusoidally:

$$y' = y - y^2 - 0.2\sin t \tag{3}$$

We can almost see the solution curves being traced out, guided along by the slope field. It appears that if the initial population is not too small, then the resulting population curve eventually looks like a sinusoid of period 2π, which is the period of the harvesting/restocking sine function. We haven't proven that this is so, but the visual evidence is strong.

The solution curves in Figure 2.4.5 result from forward or backward solving from appropriately selected initial points on the screen.

The slope field in Figure 2.4.5 suggests (as the solution curves in the figure already show) that solutions with initial values $y(0)$ slightly greater than 0.1 merge as $t \rightarrow$

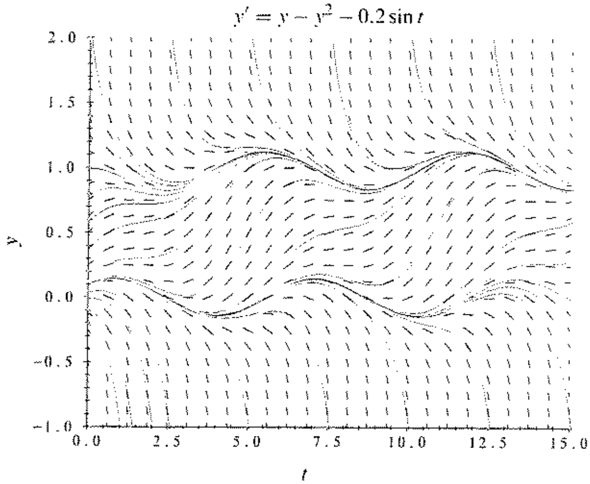


FIGURE 2.4.5 Slope field and some solution curves for ODE (3) in Example 2.4.4.

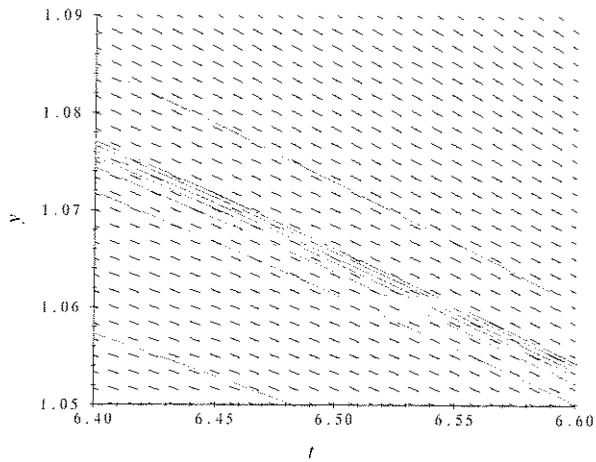
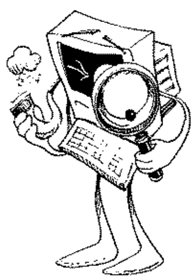


FIGURE 2.4.6 Zoom on a portion of Figure 2.4.5 centered at the point (6.50, 1.07).



$+\infty$, but solutions with $y(0)$ slightly less than 0.1 fall away and never come back. This information about solutions is *qualitative* (i.e., it describes general solution behavior), but it is not as precise as the *quantitative* behavior provided by a solution formula. In the absence of solution formulas, information about qualitative behavior often is all we need, or can obtain.

Figure 2.4.6 shows what happens when the *zoom* feature of a numerical solver is used to magnify a small region of the ty-plane. Why do the segments of the solution curves look like parallel lines in this microscopic view? It's because the rate function $y - y^2 - 0.2\sin t$ is a continuous function of t and y, and so the values of the slopes of the field lines change very little if t and y vary slightly.

Looking Back

☞ Computer tip for producing a family of curves.

We used a technique in Figures 2.4.3 and 2.4.5 that turns out to be useful in generating a family of solution curves. We selected initial points on the edges or in the interior of the rectangle defined by the computer screen and solved the IVP forward and backward in time (as appropriate) until the solution curve exits the screen. We used this technique to produce many graphs of families of solution curves in this text. Indeed, it would be difficult to produce some of these graphs using only forward IVPs. A word of caution: before using this technique, turn off the automatic scaling feature of your solver to avoid having a runaway solution curve defeat your effort.

We saw how a slope field for an ODE aids in the visualization of solution curves. Figures 2.4.1–2.4.6 suggest that we could sketch a broken-line approximation for the solution curve of an IVP by just following slope field line segments one after another beginning with the one through the initial point. This approach is the basic algorithm behind one-step numerical solvers. See the SPOTLIGHT ON APPROXIMATE NUMERICAL SOLUTIONS for some specific algorithms used in numerical solvers.

PROBLEMS

Nullclines, Slope Fields, Solution Curves. Here are two slope fields and the graphs (dashed) of nullclines. In each case, find the appropriate ODE from the list of six ODEs. Make an enlarged photocopy and sketch four distinct solution curves. Each solution curve should reach from edge to edge and cut the nullcline at least once. [*Hint*: to shorten the list of possible ODEs, consider the nature of the nullcline of each ODE. Then take a look at slope field elements above and below the nullclines.]

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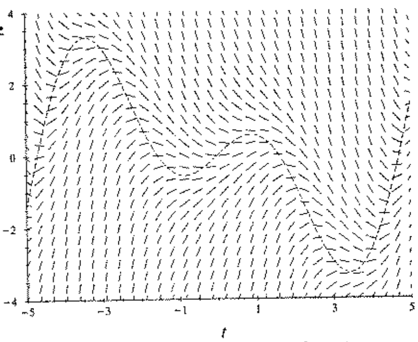
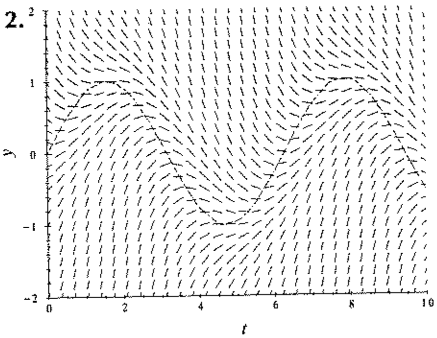
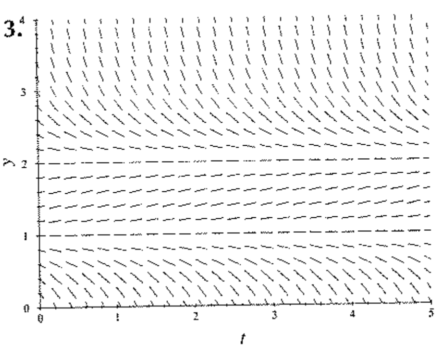
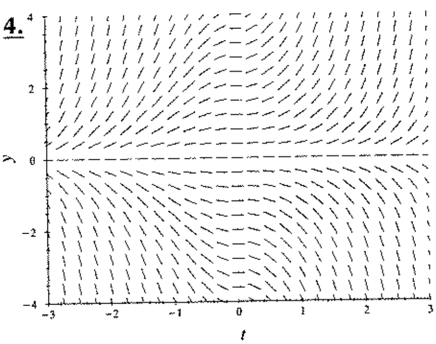


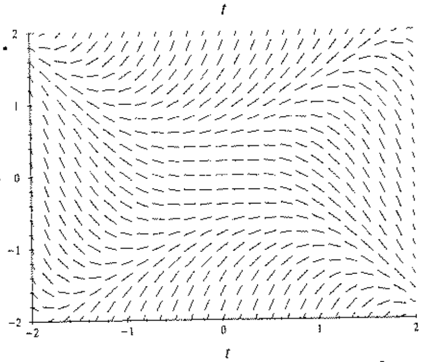
2.


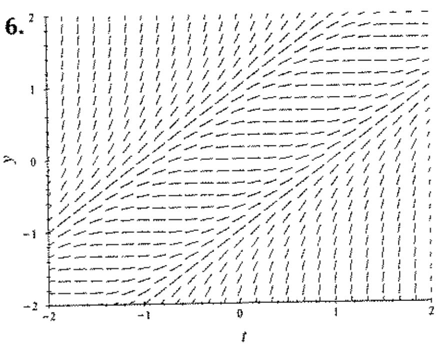
a. $y' = y - \sin t$ **b.** $y' = -y + \cos t$ **c.** $y' = y\cos t$
d. $y' = -y + \sin t$ **e.** $y' = -3y + t\cos t$ **f.** $y' = -y + t\cos t$

Match the Slope Field with an ODE. Here are four slope fields and nine ODEs. For each slope field select the ODE that determines the field. Give reasons for your choices. [*Hint*: first look for any equilibrium solution lines (the rate function must be zero on these lines). Then locate the other nullclines and look at the sign of the rate function on horizontal and vertical lines in the ty-plane.]

3.


4.


5.


6.


a. $y' = -(y-1)(y-2)$ **b.** $y' = ty$ **c.** $y' = |t|y$
d. $y' = 2 - y$ **e.** $y' = y^2 - t^2$ **f.** $y' = t^2 - y^2$
g. $y' = (y-t)^2$ **h.** $y' = y(1-y) - 2$ **i.** $y' = y^2 - t$

☞ This icon indicates that this problem is designed for the ODE Architect Tool.

Slope Fields. Enter the ODE into the **Editing Pane** of ODE Architect's solver tool and fix the scales to the indicated values as follows: depress the top button of the **Graph Mode Controls** to the right of a graphics screen and select **Scales**. In the **Graph Scales** panel click off **Auto Scale** and insert values for the **X-scale**. Repeat for the **Y-scale**. Create a slope field as follows: depress the top button of the **Graph Mode Controls** and select **Direction Fields**. Modify your slope field as follows: depress the top button of the **Graph Mode Controls** button and select **Edit**. On the **Edit 2D Graph** panel hit **Dir Field** and make your changes on the panel that comes up. Print your final slope field.

7. $y' = (1-t)y - t$; $-2 \le t \le 4$, $|y| \le 3$ **8.** $y' = y^2 - 4ty + t^2$; $|t| \le 3$, $|y| \le 5$

9. $y' = t^2/(y^2-1)$; $|t| \le 2$, $|y| \le 2$ [*Hint*: the field segments are vertical at $y = \pm 1$.]

Sketching Solution Curves. Using nullclines, slope fields, and the sign of the rate function, sketch some solution curves for the ODEs. Verify your sketch by using a numerical solver.

10. $y' = (y+3)(y-2)$. Why can't the solution curve with initial point $t_0 = 0$, $y_0 = 0$, fall below the line $y = -3$ as t increases?

11. $y' = 2t - y$. Show that $y = 2t - 2$ is a solution. As t increases, what happens to all the solution curves that don't touch the straight line $y = 2t - 2$?

12. $y' = ty - 1$. As $t \to +\infty$, what happens to the solution curve passing through the point $t = 1$, $y = 1$? Through the point $t = -1$, $y = -1$? Through the point $t = 0$, $y = 0$?

13. $y' = (1-t)y$. What is the behavior of solutions as t increases? What happens as $t \to +\infty$?

14. $y' + (\sin t)y = t\cos t$. What is the long-term behavior of every solution that passes through the y-axis, both as $t \to +\infty$ and as $t \to -\infty$?

15. $y' = y - t^2$. For what values of the constants A, B, and C is the parabola defined by $y = At^2 + Bt + C$ a solution curve. What happens as $t \to -\infty$ to every solution curve that starts out at $t = 0$ above this parabola? As $t \to +\infty$?

Nullclines, Slope Fields, and Solution Curves.

16. *Rise and Fall of Solution Curves* Imagine the rectangle R, $|t| \le 6$, $|y| \le 8$, to be filled with solution curves of the ODE $y' = -y\cos t$. Sketch the nullclines. Where are the solution curves rising? Falling? Make a rough sketch in R of the solution curves. Are the solution curves periodic; if so, what is the period? [*Hint*: find a formula for the general solution.]

17. *Slope Field* For the nonlinear ODE $y' = 3y\sin y + t$ of Figure 2.4.3, there is no known explicit solution formula, so we must turn to a slope field or to a numerical ODE solver to see how solution curves behave.

☞ See the WEB SPOTLIGHT ON SLOPE FIELDS II

(a) Plot a slope field over the rectangle, $-6 \le t \le 3$, $-2 \le y \le 4$. Sketch some solution curves suggested by the slope field. Redraw your slope field with a finer grid, and see if your curves need any adjustment. Use a numerical solver to check the accuracy of your sketches. Record your observations about the use of slope fields to sketch solution curves.

(b) Take initial points $t_0 = -6$, $1.8 \le y_0 \le 2$, solve forward 10 units of time, and try to hit the target point $(0, 2)$. Is there a difficulty? Is there another way to "win"?

18. $y' = -2y(y-3)$. Suppose that $y_0 = y(0)$ is chosen as follows: $y_0 > 3$, $0 < y_0 < 3$, $y_0 < 0$. What happens to every solution curve with $y(0) = y_0$ as t increases? As t decreases? [*Hint*: $y = 0$ and $y = 3$ are nullclines.]

19. $y' = y^2 - t$. What happens to every solution curve as t increases?

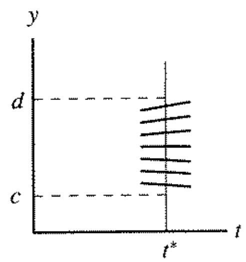
20. $y' = -(5t - 4y)/(t + y)$. Describe the behavior of the solution curves above the line $y = -t$, as t increases. What happens to solution curves below the line as t increases?

21. $y' = 1 - e^{ty}$. Explain why no solution curve with $y(0) > 0$ rises as t increases beyond $t = 0$. Why do these solution curves never fall below the t-axis?

Qualitative Analysis.

22. Show that the solution of the IVP $y' = 1 - t \sin y$, $y(1) = 1$, remains in the half-strip $t \geq 1$, $0 < y < \pi/2$, for all $t \geq 1$. Plot a direction field for $1 \leq t \leq 6$, $-0.5 \leq y \leq 2$, and the solution curve of the IVP.

23. Show that the solution of the IVP $y' = -t^2 \sin y$, $y(0) = \pi/4$, stays in the half-strip $t \geq 0$, $0 < y < \pi/2$, for all $t \geq 0$. Plot a direction field and the solution curve for $0 \leq t \leq 3$, $-1 \leq y \leq 2$.



24. *Diverging and Converging Flows* Think of the solution curves of the ODE $y' = f(t, y)$ as paths traced out by particles moving along *flow lines* of a fluid whose velocity at (t, y) is $f(t, y)$. Answer the questions below:

(a) If $\partial f/\partial y > 0$ at some $t = t^*$ and all y, $c < y < d$, explain why the flow slopes across the line $t = t^*$ diverge as indicated in the margin diagram. If $\partial f/\partial y > 0$ on some region in the ty-plane, what can be said about the solution curves in that region as t increases?

(b) Formulate an analogous property of flows in regions where $\partial f/\partial y < 0$.

(c) Illustrate the properties in **(a)** and **(b)** with the ODE $y' = 0.1(y - 3)(y^2 - 1)$. Since the ODE is autonomous, any t^* will do, so take $t^* = 0$. Use a numerical solver to plot some of the diverging or converging solution curves for $0 \leq t \leq 5$, $-2 \leq y \leq 4$.

25. *Using Slope Fields to Bound Solutions* Let $f(t, y)$ and $\partial f/\partial y$ be continuous on the whole ty-plane. Let $g(t)$ be a continuously differentiable function defined for all t.

(a) Suppose $g'(t) > f(t, g(t))$ for $t \geq T$, for some T. Also suppose that $y(t)$ is a solution of the ODE $y' = f(t, y)$ and that $y(T) < g(T)$. Show that the solution curve $y = y(t)$ always remains below the curve $y = g(t)$ for all $t \geq T$.

(b) Suppose $g'(t) < f(t, g(t))$ for $t \geq T$, for some T. Formulate and prove a property of solution curves of the ODE $y' = f(t, y)$ analogous to the property in **(a)**.

(c) Find constants k_1 and k_2 such that the solution curves of the ODE $y' = y - y^2 - 0.2 \sin t$ with initial data $y(0)$ in the interval $k_1 < y(0) < k_2$ remain in that strip for all time. Find the smallest interval with that property. Is your result consistent with Figure 2.4.5? [*Hint*: take $g(t) = k_2$ in **(a)**, and $g(t) = k_1$ in **(b)**.]

2.5 Separable Differential Equations: Planar Systems

In Sections 2.1 and 2.2 we showed how to find and interpret formulas for all solutions of first-order *linear* ODEs. Now it's time to look at some first-order nonlinear ODEs. In this section we will look at ODEs that can be written in the form $N(y)y' + M(t) = 0$. These are *separable* ODEs because we can separate the variables y and t as indicated.

Separating the Variables

As we will see later on, there are advantages to using x as an independent variable name instead of t. So let's find a solution formula for the separable ODE

$$N(y)y' + M(x) = 0 \tag{1}$$

where $y' = dy/dx$. This ODE is often written as $N(y)y' = -M(x)$ with the equality sign separating the y-terms and the x-terms. Let's suppose from now on that $N(y)$ and