Math 342A — Take Home — 10/04

1) a. $\vec{\nabla} f = z\left(\frac{-y/x^2}{1+(y/x)^2}\right)\vec{i} + z\left(\frac{1/x}{1+(y/x)^2}\right)\vec{j} + \arctan(y/x)\vec{k} = -\frac{zy}{x^2+y^2}\vec{i} + \frac{zx}{x^2+y^2}\vec{j} + \arctan(y/x)\vec{k}$

b. $\vec{\nabla}\cdot\vec{V} = y/x + ze^{yz} + \cos(xz) - xz\sin(xz)$

c. $\vec{\nabla}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\ln x & e^{yz} & z\cos(xz) \end{vmatrix} = -ye^{yz}\vec{i} + z^2\sin(xz)\vec{j} - \ln x\,\vec{k}$

d. $\vec{V}\cdot\vec{\nabla}\times\vec{\nabla} f = 0$ as $\vec{\nabla}\times\vec{\nabla} f$ always is zero

2) $\vec{R}(t) = \frac{4}{9}((1+t)^{3/2}\vec{i} + \frac{4}{9}(1-t)^{3/2}\vec{j} + \frac{t}{3}\vec{k}$, vel. $\vec{R}'(t) = \frac{2}{3}(1+t)^{1/2}\vec{i} - \frac{2}{3}(1-t)^{1/2}\vec{j} + \frac{1}{3}\vec{k}$, speed $v = \|R'(t)\| = [\frac{4}{9}(1+t) + \frac{4}{9}(1-t) + \frac{1}{9}]^{1/2} = 1 \Rightarrow \vec{T} = \vec{R}'(t)$, $\vec{a}(t) = \vec{R}''(t) = \frac{1}{3}(1+t)^{-1/2}\vec{i} + \frac{1}{3}(1-t)^{-1/2}\vec{j}$, tangential component of acceleration $a_T = \frac{dv}{dt} = 0$ as speed is constant, $\therefore a_N = [\frac{1}{9}(1+t)^{-1} + \frac{1}{9}(1-t)^{-1}]^{1/2} = \frac{\sqrt{2}}{3}(1-t^2)^{-1/2}$

$\vec{N} = \frac{\vec{a}(t)}{a_N} = \frac{1}{\sqrt{2}}\left[(1-t)^{1/2}\vec{i} + (1+t)^{1/2}\vec{j}\right]$, Since $v^2K = a_N \Rightarrow K = a_N$, $\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{2}{3}(1+t)^{1/2} & -\frac{2}{3}(1-t)^{1/2} & 1/3 \\ \frac{1}{\sqrt{2}}(1-t)^{1/2} & \frac{1}{\sqrt{2}}(1+t)^{1/2} & 0 \end{vmatrix}$

$= \frac{-1}{3\sqrt{2}}(1+t)^{1/2}\vec{i} + \frac{1}{3\sqrt{2}}(1-t)^{1/2}\vec{j} + \frac{2}{3\sqrt{2}}[1+t+1-t]\vec{k} = \frac{1}{3\sqrt{2}}\left[-(1+t)^{1/2}\vec{i} + (1-t)^{1/2}\vec{j} + 4\vec{k}\right]$

Since $s(t) = \int_a^t v\,ds = t-a \therefore ds = dt \Rightarrow \frac{d\vec{B}}{ds} = \frac{d\vec{B}}{dt} = \frac{1}{3\sqrt{2}}\left[-\frac{1}{2}(1+t)^{-1/2}\vec{i} - \frac{1}{2}(1-t)^{-1/2}\vec{j}\right] = -\frac{1}{6\sqrt{2}}\left[(1+t)^{-1/2}\vec{i} + (1-t)^{-1/2}\vec{j}\right]$

$= -\frac{1}{6(1-t^2)^{1/2}}\vec{N} \therefore \tau = \frac{1}{6}(1-t^2)^{-1/2}$

3) a. $\vec{v}(t) = \sinh t\,\vec{i} + \cosh t\,\vec{j} + \vec{k}$, $\vec{a}(t) = \cosh t\,\vec{i} + \sinh t\,\vec{j}$, $\vec{r}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cosh t & \sinh t & t \\ \cosh t & \sinh t & 0 \end{vmatrix} = -t\sinh t\,\vec{i} + t\cosh t\,\vec{j}$

$\vec{v}\cdot(\vec{r}\times\vec{a}) = -t\sinh^2 t + t\cosh^2 t = t$

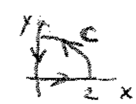
b. $L = \int_0^1 \|\vec{v}(t)\|\,dt = \int_0^1 (\sinh^2 t + \cosh^2 t + 1)^{1/2}dt = \sqrt{2}\int_0^1 \cosh t\,dt = \sqrt{2}\sinh(1)$

c. $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\sin(z) & x\sin(z) & xy\cos(z) \end{vmatrix} = 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$ $\therefore \vec{F}$ is conservative $\phi(x,y,z) = \int y\sin(z)\,dx = xy\sin(z) + h(y,z)$

$\frac{\partial\phi}{\partial y} = \frac{\partial h}{\partial y} + x\sin z = x\sin z \Rightarrow h(y,z) = k(z)$. $\frac{\partial\phi}{\partial y} = k'(z) + xy\cos(z) \Rightarrow \phi(x,y,z) = xy\sin(z)$

$W = \oint_C \vec{F}\cdot d\vec{r} = \phi(x,y,z)\Big|_{(1,0,0)}^{(\cosh 1,\,\sinh 1,\,1)} = \cosh(1)\sinh(1)\sin(1)$

4) a. $\frac{\partial g}{\partial x} \neq \frac{\partial f}{\partial y}$ $\therefore \vec{F}$ is not conservative $\oint_C \vec{F}\cdot d\vec{r} = \int_0^2 (y\cos(x) - x^2(y+1))\Big|_{y=0} dx = -\int_0^2 x^2\,dx = -\frac{x^3}{3}\Big|_0^2 = -\frac{8}{3}$



b. $\oint_C \vec{F}\cdot d\vec{R} \overset{\text{Green's Th}}{=} \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA = \iint_R [\cos(x) - \cos(x) + x^2]\,dA = \int_0^{\pi/2}\int_0^2 r^2\cos^2\theta\, r\,dr\,d\theta$

$= \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta\, \frac{r^4}{4}\Big|_0^2 = 4\left(\frac{\pi}{4}\right) = \pi$

5) a. $\vec{n} = -\vec{k}$, $\iint_{S_1} \vec{V}\cdot\vec{n}\,d\sigma = -\iint_{S_1} (z^2+1)\Big|_{z=0} dA = -4\pi$ (area of $S_1 = 4\pi$)

b. $\iint_{S_1+S_2} \vec{V}\cdot\vec{n}\,d\sigma = \iiint_M \vec{\nabla}\cdot\vec{V}\,dV = \iiint_M \left(\frac{y}{x} - \frac{2y}{2x} + 2z\right)dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^2 2\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$= 2\theta\Big|_0^{2\pi} \frac{\rho^4}{4}\Big|_0^2 \frac{\sin^2\phi}{2}\Big|_0^{\pi/2} = 8\pi$

c. Amt leaving top $= \iint_{S_2} \vec{V}\cdot\vec{n}\,d\sigma = \iint_{S_2+S_1} \vec{V}\cdot\vec{n}\,d\sigma - \iint_{S_1} \vec{V}\cdot\vec{n}\,d\sigma = 12\pi$

6\. a. $\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u^2\vec{k}$, $0 \le u \le 2$, $0 \le v \le 2\pi$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -2u^2\cos v\,\vec{i} - 2u^2\sin v\,\vec{j} + u\,\vec{k}$$

b. $$\iint_S dA = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA = \int_0^{2\pi}\int_0^2 \left[4u^4\cos^2 v + 4u^4\sin^2 v + u^2\right]^{1/2} du\,dv$$

$$= \frac{2\pi}{8}\int_0^2 8u\,(4u^2+1)^{1/2}\,du = \frac{\pi}{4}\int_1^{17} w^{1/2}\,dw = \frac{\pi}{6} w^{3/2}\Big|_1^{17} = \frac{\pi}{6}\left(17^{3/2}-1\right)$$

$w = 4u^2+1 \quad dw = 8u\,du$

c. $$\oint_C \vec{F}\cdot d\vec{R} \overset{\text{Stoke's Th}}{=} \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,d\sigma = \iint_S (-1)\,dA = -4\pi$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & x & -xze^y \end{vmatrix} = -xze^y\,\vec{i} + ze^y\,\vec{j} - \vec{k}$$

Take $S = \{(x,y,z) : x^2+y^2 \le 4,\ z=4\} \Rightarrow \vec{n} = \vec{k}$

Alternately, $x = 2\cos\theta \quad y = 2\sin\theta$

$$\oint_C \vec{F}\cdot d\vec{R} = \int_0^{2\pi} \left[2(2\sin\theta)(-2\sin\theta) + 2\cos\theta(2\cos\theta)\right] d\theta = \int_0^{2\pi}\left[-4(1-\cos 2\theta) + 2(1+\cos 2\theta)\right] d\theta = -8\pi + 4\pi = -4\pi$$

13