

1) a.  $\vec{\nabla} f = z \left( \frac{-y/x^2}{1+(y/x)^2} \right) \vec{i} + z \left( \frac{1/x}{1+(y/x)^2} \right) \vec{j} + \arctan(y/x) \vec{k} = -\frac{zy}{x^2+y^2} \vec{i} + \frac{zx}{x^2+y^2} \vec{j} + \arctan(y/x) \vec{k}$

b.  $\vec{\nabla} \cdot \vec{V} = y/x + ze^{yz} + \cos(xz) - xze \sin(xz)$

c.  $\vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y \ln x & e^{yz} & z \cos(xz) \end{vmatrix} = -ye^{yz} \vec{i} + z^2 \sin(xz) \vec{j} - \ln x \vec{k}$

d.  $\vec{V} \cdot \vec{\nabla} \times \vec{V} = 0$  as  $\vec{\nabla} \times \vec{V}$  always is zero

2)  $\vec{R}(t) = \frac{y}{9} (1+t)^{3/2} \vec{i} + \frac{y}{9} (1-t)^{3/2} \vec{j} + \frac{t}{3} \vec{k}$ , vel.  $\vec{R}'(t) = \frac{2}{3} (1+t)^{1/2} \vec{i} - \frac{2}{3} (1-t)^{1/2} \vec{j} + \frac{1}{3} \vec{k}$ , speed  $v =$

$\|\vec{R}'(t)\| = [\frac{4}{9}(1+t) + \frac{4}{9}(1-t) + \frac{1}{9}]^{1/2} = 1 \Rightarrow \vec{T} = \vec{R}'(t)$ ,  $\vec{a}(t) = \vec{R}''(t) = \frac{1}{3} (1+t)^{-1/2} \vec{i} + \frac{1}{3} (1-t)^{-1/2} \vec{j}$ , tangential component of acceleration  $a_T = \frac{d\vec{v}}{dt} = 0$  as speed is constant,  $\therefore a_N = [\frac{1}{9}(1+t)^{-1} + \frac{1}{9}(1-t)^{-1}]^{1/2} = \frac{\sqrt{2}}{3} (1-t^2)^{-1/2}$

$\vec{N} = \frac{\vec{a}(t)}{a_N} = \frac{1}{\sqrt{2}} [(1-t)^{1/2} \vec{i} + (1+t)^{1/2} \vec{j}]$ , since  $v^2 \kappa = a_N \Rightarrow \kappa = a_N$ ,  $\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{2}{3}(1+t)^{1/2} & -\frac{2}{3}(1-t)^{1/2} & \frac{1}{3} \\ \frac{1}{\sqrt{2}}(1-t)^{1/2} & \frac{1}{\sqrt{2}}(1+t)^{1/2} & 0 \end{vmatrix}$

$= \frac{-1}{3\sqrt{2}} (1+t)^{1/2} \vec{i} + \frac{1}{3\sqrt{2}} (1-t)^{1/2} \vec{j} + \frac{2}{3\sqrt{2}} [(1+t) + (1-t)] \vec{k} = \frac{1}{3\sqrt{2}} [-(1+t)^{1/2} \vec{i} + (1-t)^{1/2} \vec{j} + 4\vec{k}]$   
 Since  $s(t) = \int_0^t v ds = t - a \therefore ds = dt \Rightarrow \frac{d\vec{B}}{ds} = \frac{d\vec{B}}{dt} = \frac{1}{3\sqrt{2}} [-\frac{1}{2}(1+t)^{-1/2} \vec{i} - \frac{1}{2}(1-t)^{-1/2} \vec{j}] = -\frac{1}{6\sqrt{2}} [(1+t)^{-1/2} \vec{i} + (1-t)^{-1/2} \vec{j}]$   
 $\therefore \vec{\tau} = -\frac{1}{6(1-t^2)^{1/2}} \vec{N} \therefore \tau = \frac{1}{6} (1-t^2)^{-1/2}$

3) a.  $\vec{v}(t) = \sinh t \vec{i} + \cosh t \vec{j} + \vec{k}$ ,  $\vec{a}(t) = \cosh t \vec{i} + \sinh t \vec{j}$ ,  $\vec{r} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cosh t & \sinh t & t \\ \cosh t & \sinh t & 0 \end{vmatrix} = -t \sinh t \vec{i} + t \cosh t \vec{j}$

$\vec{v} \cdot (\vec{r} \times \vec{a}) = -t \sinh^2 t + t \cosh^2 t = t$

b.  $L = \int_0^1 \|\vec{v}(t)\| dt = \int_0^1 (\sinh^2 t + \cosh^2 t + 1)^{1/2} dt = \sqrt{2} \int_0^1 \cosh t dt = \sqrt{2} \sinh(1)$

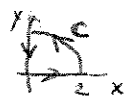
c.  $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y \sin z & x \sin z & xy \cos z \end{vmatrix} = 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$   $\therefore \vec{F}$  is conservative.  $\phi(x,y,z) = \int y \sin(z) dx = xy \sin(z) + h(y,z)$

$\frac{\partial \phi}{\partial y} = \frac{\partial h}{\partial y} + x \sin z = x \sin z \Rightarrow h(y,z) = k(z)$ .  $\frac{\partial \phi}{\partial y} = k'(z) + xy \cos(z) \Rightarrow \phi(x,y,z) = xy \sin(z)$

$W = \oint_C \vec{F} \cdot d\vec{r} = \phi(x,y,z) \Big|_{(1,0,0)}^{(\cosh 1, \sinh 1, 1)} = \cosh(1) \sinh(1) \sin(1)$

4) a.  $\frac{\partial g}{\partial x} \neq \frac{\partial f}{\partial y} \therefore \vec{F}$  is not conservative  $\oint_C \vec{F} \cdot d\vec{r} = \int_0^2 (y \cos(x) - x^2(y+1)) \Big|_{y=0}^{y=2} dx = -\int_0^2 x^2 dx = -\frac{x^3}{3} \Big|_0^2 = -\frac{8}{3}$

b.  $\oint_C \vec{F} \cdot d\vec{R} = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R [\cos(x) - \cos(x) + x^2] dA = \int_0^{2\pi} \int_0^2 r^2 \cos^2 \theta r dr d\theta$   
 $= \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} d\theta \frac{r^4}{4} \Big|_0^2 = 4 \left( \frac{\pi}{4} \right) = \pi$



5) a.  $\vec{n} = -\vec{k}$ ,  $\iint_{S_1} \vec{V} \cdot \vec{n} d\sigma = -\iint_{S_1} (z^2 + 1) \Big|_{z=0} dA = -4\pi$  (area of  $S_1 = 4\pi$ )

b.  $\iint_{S_1+S_2} \vec{V} \cdot \vec{n} d\sigma = \iiint_M \vec{\nabla} \cdot \vec{V} dV = \iiint_M \left( \frac{y}{x} - \frac{2y}{2x} + 2z \right) dV = \int_0^{2\pi} \int_0^{2\pi} \int_0^2 2 \rho \cos \phi \rho^2 \sin \phi d\rho d\theta d\phi$   
 $= 2\theta \Big|_0^{2\pi} \frac{\rho^4}{4} \Big|_0^2 \frac{\sin^2 \phi}{2} \Big|_0^{2\pi} = 8\pi$

c. Amt leaving top =  $\iint_{S_2} \vec{V} \cdot \vec{n} d\sigma = \iint_{S_2+S_1} \vec{V} \cdot \vec{n} d\sigma - \iint_{S_1} \vec{V} \cdot \vec{n} d\sigma = 12\pi$

Take Home (cont.)

⑥ a.  $\vec{r}(u, v) = u \cos v \vec{i} + u \sin v \vec{j} + u^2 \vec{k}$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2\pi$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = -2u^2 \cos v \vec{i} - 2u^2 \sin v \vec{j} + u \vec{k}$$

b.  $\iint_S dA = \iint_D |\vec{r}_u \times \vec{r}_v| dA = \int_0^{2\pi} \int_0^2 [4u^4 \cos^2 v + 4u^4 \sin^2 v + u^2]^{1/2} du dv$   
 $= \frac{2\pi}{2} \int_0^2 8u (4u^2 + 1)^{1/2} du = \frac{\pi}{4} \int_1^{17} w^{1/2} dw = \frac{\pi}{6} w^{3/2} \Big|_1^{17} = \frac{\pi}{6} (17^{3/2} - 1)$

c.  $\oint_C \vec{F} \cdot d\vec{R} \stackrel{\text{Stokes' Thm}}{=} \iint_S \nabla \times \vec{F} \cdot \vec{n} d\sigma = \iint_S (-1) dA = -4\pi$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zy & x & -xze^y \end{vmatrix} = -xze^y \vec{i} + ze^y \vec{j} - \vec{k}$$

Take  $S = \{(x, y, z) : x^2 + y^2 \leq 4, z = 4\} \Rightarrow \vec{n} = \vec{k}$

Alternately,  $x = 2 \cos \theta$   $y = 2 \sin \theta$

$$\oint_C \vec{F} \cdot d\vec{R} = \int_0^{2\pi} [2(2 \sin \theta)(-2 \sin \theta) + 2 \cos \theta(2 \cos \theta)] d\theta = \int_0^{2\pi} [-4(1 - \cos 2\theta) + 2(1 + \cos 2\theta)] d\theta = -8\pi + 4\pi = -4\pi$$