1. a. This is a separable differential equation, so

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 y \\ \int \frac{dy}{y} &= \int 3t^2 dt \\ \ln|y| &= t^3 + C \\ y(t) &= e^{t^3+C} = Ae^{t^3} \end{aligned}$$

c. This is a linear equation like Newton's Law of Cooling. We write $\frac{dy}{dt} = 3 - 2y = -2\left(y - \frac{3}{2}\right)$, so take $z(t) = y(t) - \frac{3}{2}$. Thus, $z' = -2z$, which has the solution

$$\begin{aligned} z(t) &= Ce^{-2t} = y(t) - \frac{3}{2} \\ y(t) &= Ce^{-2t} + \frac{3}{2} \end{aligned}$$

e. This is a separable differential equation, so

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 y^2 \\ \int y^{-2} dy &= \int 3t^2 dt \\ -y^{-1} &= t^3 + C \\ y(t) &= -\frac{1}{t^3 + C} \end{aligned}$$

g. This is a separable differential equation, so

$$\begin{aligned} 2ty\frac{dy}{dt} &= t^2 + 4 \\ \int 2y dy &= \int \left(t + \frac{4}{t}\right) dt \\ y^2 &= \frac{t^2}{2} + 4\ln|t| + C \\ y(t) &= \sqrt{\frac{t^2}{2} + 4\ln|t| + C} \end{aligned}$$

2. a. This is a separable differential equation, so

$$\begin{aligned} \frac{dy}{dt} &= 2ty \\ \int \frac{dy}{y} &= \int 2t dt \\ \ln|y| &= t^2 + C \\ y(t) &= e^{t^2+C} = Ae^{t^2} \end{aligned}$$

From the initial condition, $y(0) = 5 = A$, it follows that the solution is given by

$$y(t) = 5e^{t^2}.$$

c. This is a separable differential equation, so

$$\begin{aligned} (1+2y)\frac{dy}{dt} &= 2t \\ \int (1+2y)dy &= \int 2tdt \\ y + y^2 &= t^2 + C \end{aligned}$$

This gives an implicit answer for the solution into which we substitute the the initial condition, $y(2) = 0$. It follows that

$$0 + 0^2 = 2^2 + C \qquad \text{or} \qquad C = -4.$$

The implicit answer is

$$y^2(t) + y(t) = t^2 - 4,$$

which can be solved explicitly to give

$$y(t) = \frac{-1 + \sqrt{4t^2 - 15}}{2}.$$

f. This is a separable differential equation, so

$$\begin{aligned} t\frac{dy}{dt} &= 2y \\ \int \frac{dy}{y} &= \int \frac{2dt}{t} \\ \ln|y| &= 2\ln|t| + C \\ y(t) &= e^{2\ln|t|+C} = e^{\ln(t^2)}e^C = At^2 \end{aligned}$$

The initial condition, $y(1) = 4$, is substituted into this equation giving $y(1) = 4 = A(1)^2$ or $A = 4$. It follows that

$$y(t) = 4t^2.$$

g. This is a separable differential equation, so

$$\begin{aligned} \frac{dy}{dt} &= 2\cos(2t)y^2 \\ \int \frac{dy}{y^2} &= \int 2\cos(2t)dt \\ -\frac{1}{y(t)} &= \sin(2t) + C \\ y(t) &= -\frac{1}{\sin(2t) + C} \end{aligned}$$

The initial condition, $y(0) = 1$, is substituted into this equation giving $y(0) = 1 = -\frac{1}{C}$ or $C = -1$. It follows that

$$y(t) = \frac{1}{1 - \sin(2t)}.$$

4. a. Begin by separating the variables in this differential equation. The result is as follows:

$$\begin{aligned}
\frac{dV}{dt} &= 0.04V^{3/4} \\
\int V^{-3/4}dV &= \int 0.04dt \\
4V^{1/4} &= 0.04t + C \\
V(t) &= \left(0.01t + \frac{C}{4}\right)^4
\end{aligned}$$

The initial condition, $V(0) = 1$, is substituted into this equation giving $V(0) = 1 = \left(\frac{C}{4}\right)^4$ or $C = 4$. It follows that

$$V(t) = (0.01t + 1)^4.$$

b. For the cell to double its volume, we solve

$$\begin{aligned}
V(t) &= (0.01t + 1)^4 = 2 \\
(0.01t + 1) &= 2^{1/4} \\
t &= 100\left(2^{1/4} - 1\right) \simeq 18.92.
\end{aligned}$$

Thus, it takes about 18.92 time units for the cell to double its volume.

5. a. The differential equation described in this problem is given by

$$\frac{dV}{dt} = kV^{2/3}.$$

The general solution is found by separating variables to give

$$\begin{aligned}
\int V^{-2/3}dV &= \int kdt \\
3V^{1/3} &= kt + C \\
V(t) &= \left(\frac{kt + C}{3}\right)^3.
\end{aligned}$$

b. If $V(0) = 1$, then $V(0) = \left(\frac{C}{3}\right)^3 = 1$ implies that $C = 3$. Thus, the solution for the growth of the raindrop is

$$V(t) = \left(\frac{0.1\,t + 3}{3}\right)^3.$$

For this solution to grow to 8 units,

$$V(t) = \left(\frac{0.1\,t + 3}{3}\right)^3 = 8 \qquad \text{or} \qquad \frac{0.1\,t + 3}{3} = 2.$$

It follows that $t = 30$ time units.

7. a. Since $N(t) = k\pi x^2(t)$, $x(t) = \sqrt{N(t)/k\pi}$. It follows that $C(t)$ can be written

$$C(t) = 2\sqrt{\frac{\pi N(t)}{k}}.$$

b. The assumption is that the rate of spread of the disease (which is the change in number of diseased trees) is proportional to the circumference of the infected region. Since $C(t)$ is the circumference (and we chose q to be the proportionality factor), the differential equation is given by

$$\frac{dN}{dt} = qC, \qquad N(0) = 1,$$

assuming we start with one infected tree in the middle of the grove.

c. With the model for the spread of a disease in an orchard, we apply the separation of variables technique to the initial value problem

$$\frac{dN}{dt} = 2q\sqrt{\frac{\pi}{k}}N^{1/2}, \qquad N(0) = 1.$$

It follows that

$$\begin{aligned} \int N^{-1/2} dV &= \int 2q\sqrt{\frac{\pi}{k}}dt \\ 2N^{1/2} &= 2q\sqrt{\frac{\pi}{k}}t + C \end{aligned}$$

$$N(t) = \left(q\sqrt{\frac{\pi}{k}}t + \frac{C}{2}\right)^2$$

The initial condition, $N(0) = 1$, is substituted into this equation giving $N(0) = 1 = \left(\frac{C}{2}\right)^2$ or $C = 2$. It follows that

$$N(t) = \left(q\sqrt{\frac{\pi}{k}}t + 1\right)^2.$$

8. a. The solution to the Malthusian growth equation is $Y(t) = 2000\,e^{0.08t}$. It doubles when $Y(t) = 4000$, so $4000 = 2000\,e^{0.08t}$ or $e^{0.08t} = 2$. Thus, $0.08t = \ln(2)$ or $t = 12.5\ln(2) = 8.66$ hr.

b. This is a separable equation, so

$$\begin{aligned} \frac{dY}{dt} &= (0.08 - 0.002\,t)Y \\ \int \frac{dY}{Y} &= \int (0.08 - 0.002\,t)dt \\ \ln|Y(t)| &= 0.08\,t - 0.001\,t^2 + C \\ Y(t) &= \exp 0.08\,t - 0.001\,t^2 + C = A\,e^{0.08\,t - 0.001\,t^2} \end{aligned}$$

The initial condition gives $Y(0) = 2000 = A$. It follows that

$$Y(t) = 2000\,e^{0.08\,t - 0.001\,t^2}.$$

c. The maximum occurs when the $\frac{dY}{dt} = 0$, which is true when $0.08 - 0.002\,t = 0$ or $t = 40$. From the equation above, $Y(40) = 2000\,e^{0.08(40)-0.001(40)^2} = 2000\,e^{1.6} \simeq 9906$. The population is 2000 when the exponent of the solution is zero, so $0.08\,t - 0.001\,t^2 = 0$ when $t = 0$ or 80 hrs. Thus, the population returns to 2000 after 80 hrs. (Short solutions have the graph.)

10. a. For convenience, let 1941 correspond to $t = 0$ and define $M(t)$ to be the population of India. The Malthusian growth model is

$$\frac{dM}{dt} = kM, \qquad M(0) = 319 \text{ (million)}.$$

The solution to this is $M(t) = 319e^{kt}$. Since the population in 1961 ($t = 20$) is 439 million, the Malthusian growth model gives $M(20) = 319e^{20k} = 439$, so $k = \frac{1}{20}\ln\left(\frac{439}{319}\right) \simeq 0.015965$. The population from this model for 1951 is given by $M(10) = 319e^{0.015965(10)} = 374.2$ million. The percent error is $100 \times \frac{|374.2-361|}{361} = 3.7\%$.

b. We solve the separable differential equation

$$\begin{aligned} \frac{dP}{dt} &= (at+b)p \\ \int \frac{dP}{P} &= \int (at+b)dt \\ \ln|P| &= \frac{at^2}{2} + bt + C \\ P(t) &= exp\left(\frac{at^2}{2} + bt + C\right) = A \cdot exp\left(\frac{at^2}{2} + bt\right). \end{aligned}$$

From the initial data, $P(0) = 319$, it follows that $A = 319$. From the population data in 1951 and 1961, we have

$$P(10) = 361 = 319e^{50a+10b} \quad \text{and} \quad P(20) = 439 = 319e^{200a+20b}.$$

Thus,

$$e^{50a+10b} = \frac{361}{319} = 1.13166 \quad \text{and} \quad e^{200a+20b} = \frac{439}{319} = 1.3762.$$

Taking logarithms gives

$$50a + 10b = \ln(1.13166) = 0.123687 \quad \text{and} \quad 200a + 20b = \ln(1.3762) = 0.319308.$$

Multiply the first equation by -2 and add it to the second equation to get $100a = 0.319308 - 0.247373$ or $a = 0.00071935$. Similarly, multiply the first equation by -4 and add it to the second equation to obtain $-20b = 0.319308 - 0.494747$ or $b = 0.0087720$. Thus, the nonautonomous Malthusian growth model becomes

$$P(t) = 319e^{0.00035967t^2+0.0087720t}.$$

c. The Malthusian growth model gives the population in 1991 as $M(50) = 708.7$ million, while the nonautonomous Malthusian growth model gives $P(50) = 1,215.6$ million. The percent error from the actual population of 846 million for the Malthusian growth model is 16.2%, while the percent error for the nonautonomous Malthusian growth model is 43.7%. So in this case, the Malthusian growth model is the better model. See the short solutions for the graph.