

1. a. The solution to $\frac{dy}{dt} = -0.2y$, $y(0) = 8$ is $y(t) = 8e^{-0.2t}$ by simple recognition.

b. The differential equation is given by $\frac{dx}{dt} = 3 - 0.1x = -0.1(x - 30)$. We make the substitution $z(t) = x(t) - 30$ or $z(0) = 4 - 30 = -26$, since $x(0) = 4$. The modified differential equation is $z' = -0.1z$, which has the solution $z(t) = -26e^{-0.1t} = x(t) - 30$. It follows that $x(t) = 30 - 26e^{-0.1t}$.

c. The differential equation is given by $\frac{dw}{dt} = 0.02w + 4 = 0.02(w + 200)$. We make the substitution $z(t) = w(t) + 200$ or $z(0) = 2 + 200 = 202$, since $w(0) = 2$. The modified differential equation is $z' = 0.02z$, which has the solution $z(t) = 202e^{0.02t} = w(t) + 200$. It follows that $w(t) = 202e^{0.02t} - 200$.

d. The solution to $\frac{dh}{dx} = -\frac{h}{5}$, $h(0) = 50$ is $h(x) = 50e^{x/5}$ by simple recognition.

2. a. For the function, $f(x) = x^3 - 3x^2 - 1$, the y -intercept is $(0, -1)$. The derivative is $f'(x) = 3x^2 - 6x = 3x(x - 2)$, which is zero at $x = 0$ and $x = 2$. Since $f(2) = -5$, there is a maximum at $(0, -1)$ and a minimum at $(2, -5)$.

b. The Newton's Method formula is given by

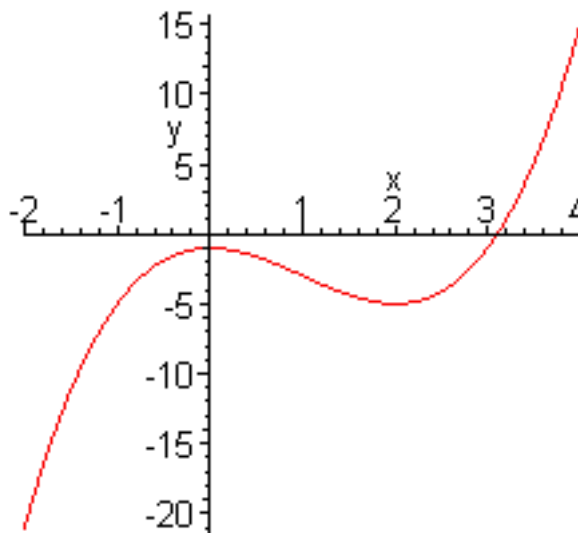
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n^2 - 1}{3x_n^2 - 6x_n}.$$

Starting with $x_0 = 3$, we have

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{-1}{9} = 3.1111 \\ x_2 &= 3.1111 - \frac{f(3.1111)}{f'(3.1111)} = 3.1111 - \frac{0.075448}{10.37037} = 3.1038 \end{aligned}$$

which is accurate to 5 significant figures to the actual x -intercept.

c. Below is a graph of $f(x)$.



3. a. For the function, $f(x) = 2x - e^x + 1$, the y -intercept is $(0, 0)$ (also, an x -intercept). The derivative is $f'(x) = 2 - e^x$, which is zero when $e^x = 2$ or $x = \ln(2) \simeq 0.693147$. Since $f(\ln(2)) = 2 \ln(2) - 1 \simeq 0.38629$, there is a maximum at $(\ln(2), 2 \ln(2) - 1) \simeq (0.693147, 0.38629)$.

b. The Newton's Method formula is given by

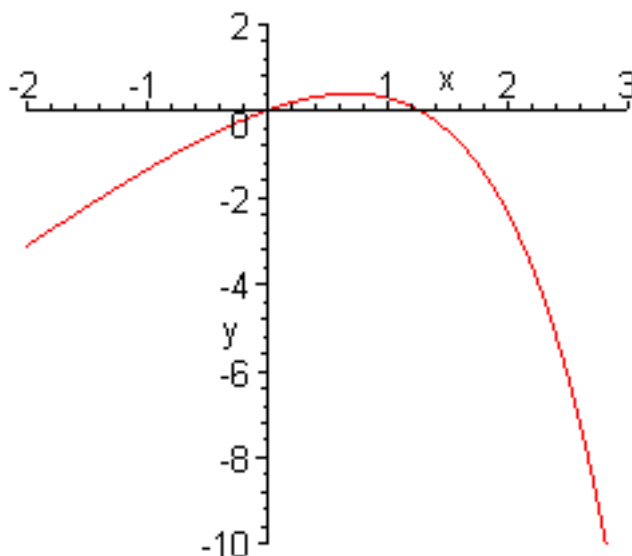
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n - e^{x_n} + 1}{2 - e^{x_n}}.$$

Starting with $x_0 = 1$, we have

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{3 - e}{2 - e} = 1.39221 \\ x_2 &= 1.3922 - \frac{f(1.3922)}{f'(1.3922)} = 1.3922 - \frac{-0.239315}{-2.02374} = 1.27396. \end{aligned}$$

The actual x -intercept is $x = 1.25643$.

c. Below is a graph of $f(x)$.



4. a. The differential equation for this culture is $B'(t) = rB(t)$, $B(0) = 1000$, which has the solution

$$B(t) = 1000e^{rt}.$$

Since $B(2) = 3000$, we have $1000e^{2r} = 3000$, so $e^{2r} = 3$ or $r = \frac{\ln(3)}{2} \simeq 0.5493 \text{ hr}^{-1}$. With this value of r , we solve $1000e^{rt_d} = 2000$, so $e^{rt_d} = 2$. Thus, the doubling time is given by $t_d = \frac{\ln(2)}{r} = \frac{2 \ln(2)}{\ln(3)} \simeq 1.262 \text{ hr}$.

b. The mutant population satisfies $M(t) = e^{0.7t}$. Solving $M(t_d) = 2 = e^{0.7t_d}$, we have the doubling time $t_d = \frac{\ln(2)}{0.7} \simeq 0.9902 \text{ hr}$. The populations $B(t)$ and $M(t)$ are equal when $1000e^{rt} = e^{0.7t}$ with $r = 0.5493$. So $e^{(0.7-r)t} = 1000$ or

$$t = \frac{\ln(1000)}{0.7 - 0.5493} \simeq 45.84 \text{ hr}.$$

5. a. For the differential equation, $\frac{dy}{dt} = y + 2$ with $y(0) = 3$ and $h = 0.25$, the Euler's formula is

$$y_{n+1} = y_n + h(y_n + 2) = y_n + 0.25(y_n + 2).$$

Iterating this, we create a table

$t_0 = 0$	$y_0 = 3$
$t_1 = 0.25$	$y_1 = y_0 + 0.25(y_0 + 2) = 3 + 0.25(3 + 2) = 4.25$
$t_2 = 0.5$	$y_2 = 4.25 + 0.25(4.25 + 2) = 5.8125$
$t_3 = 0.75$	$y_3 = 4.25 + 0.25(4.25 + 2) = 7.7656$
$t_4 = 1.0$	$y_4 = 7.7656 + 0.25(7.7656 + 2) = 10.2070$

Thus, the approximate the solution at $t = 1$ is $y_4 \simeq y(1) = 10.2070$.

b. Make a substitution, $z(t) = y(t) + 2$, so solve the initial value problem

$$z' = z, \quad z(0) = y(0) + 2 = 5.$$

This has the solution $z(t) = 5e^t = y(t) + 2$, so $y(t) = 5e^t - 2$. It follows that $y(1) = 5e - 2 \simeq 11.5914$. The error between the actual and Euler's solution is

$$100 \frac{|y(1) - y_4|}{y(1)} = 100 \frac{|11.5914 - 10.2070|}{11.5914} = 11.94\%.$$

6. a. The solution giving the population in millions is given by

$$P(t) = 50.2e^{rt},$$

where t is in years after 1880. From the population in 1890, we have $62.9 = 50.2e^{10r}$ or $e^{10r} = 1.2530$. Thus, $r = 0.02255$, we find the time until the population doubles. So $100.4 = 50.2e^{rt}$ or $t = \ln(2)/r \simeq 30.7$, which suggests that the population of the U.S. doubles from 1880 around 1911 if the rate of growth stayed the same.

b. The model predicts that the population in 1900 is

$$P(20) = 50.2e^{20r} \simeq 78.8.$$

The error between the model and the actual population is

$$100 \frac{|P(20) - 76.0|}{76.0} = 100 \frac{|78.8 - 76.0|}{76.0} = 3.7\%.$$

7. a. The solution to this differential equation is $R(t) = 10e^{-0.05t}$. The half-life satisfies $5 = 10e^{-0.05t}$, so $e^{0.05t} = 2$ or $t = 20 \ln(2) \simeq 13.86$.

b. For the differential equation, $\frac{dR}{dt} = -0.05R + 0.2e^{-0.01t}$ with $R(0) = 10$ and $h = 1$, the Euler's formula is

$$R_{n+1} = R_n + h(-0.05R_n + 0.2e^{-0.01t_n}) = R_n - 0.05R_n + 0.2e^{-0.01t_n}.$$

Iterating this, we create a table

$t_0 = 0$	$R_0 = 10$
$t_1 = 1$	$R_1 = R_0 - 0.05R_0 + 0.2e^{-0.01t_0} = 10 - 0.5 + 0.2 = 9.7$
$t_2 = 2$	$R_2 = R_1 - 0.05R_1 + 0.2e^{-0.01t_1} = 9.7 - 0.485 + 0.198 = 9.413$
$t_3 = 3$	$R_3 = R_2 - 0.05R_2 + 0.2e^{-0.01t_2} = 9.413 - 0.471 + 0.096 = 9.138$

Thus, the approximate the solution at $t = 3$ is $R_3 \simeq R(3) = 9.138$.

c. Consider $R(t) = 5e^{-0.05t} + 5e^{-0.01t}$, then $R'(t) = -0.25e^{-0.05t} - 0.05e^{-0.01t}$. But $-0.05R(t) + 0.2e^{-0.01t} = -0.05(5e^{-0.05t} + 5e^{-0.01t}) + 0.2e^{-0.01t} = -0.25e^{-0.05t} - 0.05e^{-0.01t}$, so the second choice (ii) is the solution to the radioactive decay problem with another input. The correct solution at $t = 3$ is $R(3) = 9.15576$. The percent error between the correct solution and the Euler solution

$$100 \frac{|R(3) - R_3|}{R(3)} = 100 \frac{|9.15576 - 9.138|}{9.15576} = 0.19\%.$$

8. a. The solution to the white lead problem is $P(t) = 10e^{-kt}$, where $t = 0$ represents 1970. From the data at 1975, we have $8.5 = 10e^{-5k}$ or $e^{5k} = 10/8.5 = 1.17647$. Thus, $k = 0.032504 \text{ yr}^{-1}$. To find the half-life, we compute $5 = 10e^{-kt}$, so $t = \ln(2)/k = 21.33 \text{ yr}$ is the half-life of lead-210.

b. The differential equation can be written $P' = -k(P - r/k)$, so we make the substitution $z(t) = P(t) - r/k$. This leaves the initial value problem

$$z' = -kz, \quad z(0) = P(0) - r/k = 10 - r/k,$$

which has the solution $z(t) = (P(0) - r/k)e^{-kt} = P(t) - r/k$. Thus, the solution is

$$P(t) = \left(10 - \frac{r}{k}\right) e^{-kt} + \frac{r}{k} = 2.3086e^{-kt} + 7.6914,$$

where $k = 0.032504$. In the limit,

$$\lim_{t \rightarrow \infty} P(t) = 7.6914 \text{ disintegrations per minute of } ^{210}\text{Pb}.$$

9. a. The differential equation describing the temperature of the tea satisfies

$$H' = -k(H - 21), \quad H(0) = 85 \text{ and } H(5) = 81.$$

Make the substitution $z(t) = H(t) - 21$, which gives the differential equation

$$z' = -kz, \quad z(0) = H(0) - 21 = 64.$$

The solution becomes $z(t) = 64e^{-kt} = H(t) - 21$ or

$$H(t) = 64e^{-kt} + 21.$$

To find k , we solve $H(5) = 81 = 64e^{-5k} + 21$ or $e^{5k} = 64/60 = 1.0667$. Thus, $k = 0.012908 \text{ min}^{-1}$. The water was at boiling point when $64e^{-kt} + 21 = 100$ or $e^{-kt} = 79/64$. It follows that $t = -\ln(79/64)/k = -16.3 \text{ min}$. This means that the talk went 16.3 min over its scheduled ending.

b. To obtain a temperature of at least 93°C , then we need to find the time that satisfies $H(t) = 93 = 64e^{-5k} + 21$, so $e^{-kt} = 72/64 = 1.125$. Solving for t gives $t = -\ln(72/64)/k = -9.125$ min. It follows that you must arrive at the hot water within $16.3 - 9.1 = 7.2$ min of the scheduled end of the talks.

10. a. Substituting the parameters into the differential equation gives

$$c' = \frac{1}{10^6}(22000 - 2000c) = -0.002(c - 11).$$

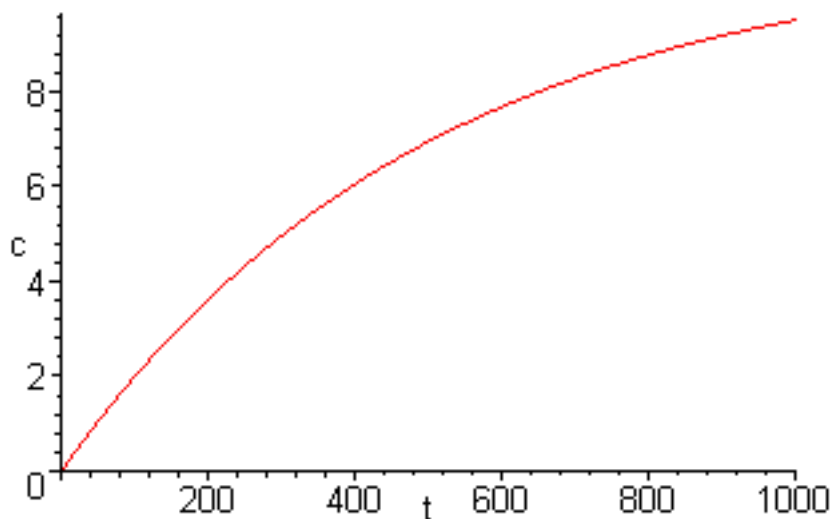
We make the substitution $z(t) = c(t) - 11$, which gives the initial value problem $z' = -0.002z$ with $z(0) = c(0) - 11 = -11$. The solution of this differential equation is $z(t) = -11e^{-0.002t} = c(t) - 11$, so

$$c(t) = 11 - 11e^{-0.002t}.$$

b. Solve the equation $c(t) = 11 - 11e^{-0.002t} = 5$, so $e^{0.002t} = 11/6$ or $t = 500 \ln(11/6) = 303.1$ days. The limiting concentration

$$\lim_{t \rightarrow \infty} c(t) = 11.$$

The graph is below.



11. a. Substituting the parameters into the differential equation gives

$$\frac{dc(t)}{dt} = \frac{200}{10000}(10 - c) = -0.02(c - 10).$$

We make the substitution $z(t) = c(t) - 10$, which gives the initial value problem $z' = -0.02z$ with $z(0) = c(0) - 10 = -10$. The solution of this differential equation is $z(t) = -10e^{-0.02t} = c(t) - 10$, so

$$c(t) = 10 - 10e^{-0.02t}.$$

We solve the equation $c(t) = 10 - 10e^{-0.02t} = 2$, so $e^{0.02t} = 10/8$ or $t = 50 \ln(10/8) = 11.2$ days.

b. The Euler's formula with the linearly increasing pollutant level is given by

$$c_{n+1} = c_n + h(0.02(10 + 0.1t_n - c_n)) = c_n + 0.2 + 0.002t_n - 0.02c_n,$$

with $h = 1$. Iterating this, we create a table

$t_0 = 0$	$c_0 = 0$
$t_1 = 1$	$c_1 = c_0 + 0.2 + 0.002t_0 - 0.02c_0 = 0.2$
$t_2 = 2$	$c_2 = c_1 + 0.2 + 0.002t_1 - 0.02c_1 = 0.398$

The approximate solution is $c_2 = 0.398$ ppb.