1. Solve the following differential equations:

a. $\dfrac{dy}{dt} = -0.2y, \quad y(0) = 8.$

b. $\dfrac{dx}{dt} = 3 - 0.1x, \quad x(0) = 4.$

c. $\dfrac{dw}{dt} = 0.02w + 4, \quad w(0) = 2.$

d. $\dfrac{dh}{dx} = -\dfrac{h}{5}, \quad h(0) = 50.$

2. Consider the function

$$f(x) = x^3 - 3x^2 - 1.$$

a. Find the y-intercept, then determine all extrema, giving both the x and y values.

b. Use Newton's Method to approximate the value of the x-intercept. Start with $x_0 = 3$ and perform two iterations. (Give 5 significant figures for each iteration.)

c. Use the information above to sketch the graph of $f(x)$.

3. Consider the function

$$f(x) = 2x - e^x + 1.$$

a. Find the y-intercept, then determine all extrema, giving both the x and y values.

b. Use Newton's Method to approximate the value of the x-intercept near $x = 1$. Start with $x_0 = 1$ and perform two iterations. (Give 6 significant figures for each iteration.)

c. Use the information above to sketch the graph of $f(x)$.

4. a. Initially, there are 1000 bacteria in a particular culture. If this culture is growing according to the Malthusian growth law and two hours later there are 3000 bacteria, then write a differential equation describing the growth of these bacteria, $B(t)$, solve the equation, and find the doubling time for this culture.

b. Suppose that a single mutant cell, $M(t)$, enters the culture (at $t = 0$) and satisfies the differential equation

$$M'(t) = 0.7M(t).$$

Solve this differential equation, find the doubling time of this mutant bacteria, then determine how long until the populations $B(t)$ and $M(t)$ are equal.

5. Consider the differential equation:

$$\frac{dy}{dt} = y + 2, \qquad y(0) = 3.$$

a. Use Euler's method with $h = 0.25$ to approximate the solution at $t = 1$.

b. Find the solution to this differential equation and evaluate $y(1)$. Determine the percent error at $t = 1$.

6. The population of the United States was about 50.2 million in 1880 and 62.9 million in 1890. Let 1880 be represented by $P(0)$ and assume that its population is growing according to the Malthusian growth law,

$$\frac{dP(t)}{dt} = rP(t),$$

where t is in years.

a. Use the data above to find the growth rate r, then solve the differential equation above. Determine how long until the U. S. population doubled from its 1880 level according to this model.

b. Predict the population in the year 1900. The actual population was about 76.0 million. What is the error between the model and the actual census data?

7. a. A radioactive substance satisfies the differential equation

$$\frac{dR}{dt} = -0.05R, \qquad R(0) = 10.$$

Find the solution of this differential equation and determine the half-life of this radioactive substance.

b. Radioactive elements are often the products of the decay of another radioactive element. A differential equation describing this situation is given by the following:

$$\frac{dR}{dt} = -0.05R + 0.2e^{-0.01t}, \qquad R(0) = 10.$$

Use Euler's method with a stepsize of $h = 1$ to find the approximate solution at $t = 3$.

c. The solution to the problem in Part b is one of the following choices:

(i). $R(t) = 8e^{-0.05t} + 2e^{-0.01t}$ (ii). $R(t) = 5e^{-0.05t} + 5e^{-0.01t}$

Select the correct solution and verify your answer. Use this answer to determine the percent error between your Euler's approximate solution and the actual solution at $t = 3$.

8. a. White lead is a pigment found in oil paints and can be used to detect art forgeries. In the absence of radium-226, lead-210 undergoes standard radioactive decay,

$$P' = -kP.$$

Suppose that a sample from a painting has 10 disintegrations per minute in 1970 and then shows 8.5 disintegrations per minute in 1975. Find the half-life of lead-210 and give the value of k.

b. When there are impurities caused by radium-226 (which has a very long half-life), the differential equation for radioactive decay is modified to

$$P' = -kP + r,$$

where $r = 0.25$ is source input from the radium-226 and k is from Part a. Solve this differential equation and determine the limit of P (disintegrations per minute of lead-210) as $t \to \infty$.

9. You are attending a conference, and the talks are going past the coffee break time. You really need a cup of tea (not liking coffee) to keep awake for the next set of talks. The refreshments are in a room that has a constant temperature of 21°C, and you find that the hot water is only 85°C. Five minutes later, the hot water is only 81°C.

a. Assume that the container of water satisfies Newton's law of cooling. ($H' = -k(H - T_e)$, where T_e is the environmental temperature.) If it was placed out when the talks were supposed to end with boiling water (water at 100°C), then how many minutes beyond the scheduled time did the talks go? (Hint: If $H(t)$ is the temperature, then use $H(0) = 85$ and $H(5) = 81$ to find the cooling constant k in Newton's law of cooling, then find when $H(t) = 100$.)

b. If tea needs water that is at least 93°C to give you enough caffeine for the next set of talks, then how long after the scheduled end of the talks can you wait?

10. a. An initially clean lake ($c(0) = 0$) concentrates pollution from an incoming stream because of evaporative loss of water of 200 m^3/day. The well-mixed lake has a stream flowing in at a rate of $f_1 = 2200$ m^3/day with a pesticide concentration of of $Q = 10$ ppb. The lake maintains a constant volume of $V = 10^6$ m^3 by having a stream leaving with a flow of $f_2 = 2000$ m^3/day. You are given that the differential equation describing the concentration of pesticide in the lake is given by

$$c' = \frac{1}{V}(f_1 Q - f_2 c).$$

Solve this differential equation.

b. Determine how long until the lake has a concentration of 5 ppb of pesticide. Also, find the limiting concentration of pesticide. Sketch a graph of the solution.

11. This problem examines pollution entering a well-mixed lake. The concentration, $c(t)$, of the pollutant in the lake with volume V maintained by a river flowing in and out at a rate f satisfies the differential equation

$$\frac{dc(t)}{dt} = \frac{f}{V}(Q(t) - c(t)),$$

where $Q(t)$ is the concentration of pollutant entering the lake from the river.

a. Assume $c(0) = 0$, $V = 10{,}000 \text{ m}^3$, $f = 200 \text{ m}^3/\text{day}$, and $Q(t) = 10$ ppb. Solve this differential equation and find how long it takes for the lake to have a concentration of 2 ppb of pollutant.

b. Suppose that the level of pollutant is increasing linearly in the river, so $Q(t) = 10 + 0.1t$. Use Euler's method with $h = 1$ and 2 steps to approximate the solution at $t = 2$.