May 23, 2001 Math 121a Name ______________________________

1. (21pts) Differentiate the following: (**Do NOT simplify!**)

a. $f(x) = 2x^4 + \dfrac{2}{e^{2x}} - (x^2 - \ln(x))^5 - \dfrac{4x^3}{x + e^{-x}}$,

b. $g(x) = x^2 e^{-3x} + \ln(x^2) - \dfrac{5}{(x+1)^3} + 4e^{-x^2}$.

2. (22pts) A number of trees from the Alleghany National Forest were measured to provide information about the quantity of wood in the forest. One tree had had a diameter (d) of 11 inches and produced a volume (V) of 18.2 board feet. Another tree measured 16 inches in diameter and had a volume of 38.2 board feet. (Give all answers to **4 significant figures**.)

a. A linear model is given by $V = md + b$ for some constants m and b. Find the constants m and b and sketch a graph of this model. Use this model to predict the number of board feet in a tree that has a diameter of 15 inches. Also, find the diameter of a tree that produces 30.7 board feet.

b. If the relationship between the diameter and volume of wood for the trees in this forest satisfies a power law of the form

$$V = kd^a,$$

Find the constants k and a and sketch a graph of this model. Use this model to predict the number of board feet in a tree that has a diameter of 15 inches. Also, find the diameter of a tree that produces 30.7 board feet.

c. Which model provides the better estimate? Give a brief reason for your choice. Give a brief explanation for the power that you obtained in Part b.

3. (24pts) For each of the following functions, find all x and y-intercepts and any asymptotes, if they exist. Find the derivative of the functions, then determine any maxima or minima. Give both the x and y values. Finally, sketch the graphs of the functions.

a. $y = (x-3)e^x$, b. $y = -2x - \dfrac{8}{x}$

4. (15pts) a. A new polymer capsule with a particular hormone embedded inside is injected under the skin of a patient, delivering the hormone to the body in a time released manner. Suppose that the a concentration of the hormone in the body ($h(t)$ in μg/l) released from this device satisfies the equation:

$$h(t) = 10(e^{-0.02t} - e^{-0.5t}),$$

with t in days. What is the initial concentration of the hormone, $h(0)$? Find when the hormone reaches its maximum concentration and determine what its maximum concentration is. Sketch a graph of the hormone concentration. List any horizontal asymptotes.

b. To monitor the effectiveness of the polymer capsule, measurements of the concentration of the hormone are gathered at several times after the implantation. Below is a table with several measurements:

t	1	10	30
$h(t)$	3.5	8.2	5.7

Find the percent error between the model and the measurement at $t = 30$. Find the sum of square errors between the model and the data.

5. (14pts) a. A culture of bacteria satisfies the Malthusian growth equation

$$P_{n+1} = 1.025 P_n, \qquad P_n = 4000,$$

where n is in minutes. Solve this growth equation and determine how long it takes for this culture to double.

b. Another culture of bacteria satisfies a similar Malthusian growth law. Suppose that this culture doubles in 25 min and starts with 1000 bacteria. Find the general solution for this culture and determine how long until the population of this bacteria is the same as the original culture from Part a.

6. (17pts) Assume that the growth rate of a population P satisfies

$$g(P) = 0.04P(1 - 0.0002P).$$

The discrete logistic growth model for this population is given by:

$$P_{n+1} = P_n + g(P_n).$$

a. Let $P_0 = 1000$ and compute P_1 and P_2.

b. Find the populations when the growth rate $g(P)$ is zero and when the growth rate is at a maximum. Sketch the graph of $g(P)$.

c. The updating function for this logistic growth model is $F(P) = P + g(P)$. Find the equilibria for the logistic growth model above. Differentiate $F(P)$ and evaluate $F'(P)$ at the equilibria. What does this say about the stability of the equilibria?

7. (12pts) a. An impala is migrating across a field that has been fenced with a 180 cm fence. To escape it needs to jump this fence. Assume that the impala jumps the fence with just enough vertical velocity, v_0 to clear it. If the height (in cm) of the impala is given by

$$h(t) = v_0 t - 490t^2,$$

then find the velocity $v(t) = h'(t)$ of the impala at any time (in sec), $t \geq 0$, before hitting the ground.

b. Find when the velocity is equal to zero in terms of v_0. This is the time at the maximum height. Since the impala is 180 cm in the air at this time, use the equation for the height, $h(t)$ to compute the initial velocity, v_0, with which the impala must launch itself to clear the fence.

c. Substitute the initial velocity into the height equation, and determine how long the impala is in the air, when jumping over the fence.

8. (14pts) The atmosphere contains about 5 ppm of Helium (He), an inert gas. Our model for breathing He shows that the concentration of He in the lungs, c_n after n breaths is given by the discrete dynamical model

$$c_{n+1} = B(c_n) = (1 - q)c_n + q\gamma,$$

where q is the fraction of air exchanged in a normal breath and $\gamma = 5$ is the atmospheric concentration of He.

a. Suppose that a subject initially has a concentration of $c_0 = 100$ ppm of He in the lungs. In the first breath, the measured concentration, $c_1 = 84.8$ ppm of He. Use this information to find the fraction of air exchanged q, then determine the concentration of He in the second breath c_2.

b. Find the equilibria for this model and determine the behavior of the solution to this model near these equilibria.

c. Sketch a graph of the updating function $B(c)$ and the identity map, showing where they intersect and the B-intercept.

9. (22pts) Hassell's model is often used for the dynamics of insect populations with discrete generations. Suppose that a population of insects satisfies the discrete model

$$P_{n+1} = H(P_n) = \frac{16P_n}{(1 + 0.005P_n)^2}.$$

a. Suppose $P_0 = 200$ and find the populations in the next two generations, P_1 and P_2.

b. Find the intercepts, extrema, and any asymptotes for $H(P)$ with $P \geq 0$. Sketch a graph of $H(P)$.

c. Find all equilibria for the model above and determine the stability of those equilibria.

10. (19pts) In fishery management, it is important to know how much fishing can be done without severely harming the population of fish. A modification of Ricker's model that includes fishing is given by the model:

$$P_{n+1} = R(P_n) = 5P_n e^{-0.001P_n} - hP_n,$$

where h is the intensity of harvesting fish.

a. Let $h = 0.5$ and $P_0 = 100$, then find P_1 and P_2.

b. With $h = 0.5$, find all equilibria for this model. Find the derivative of the updating function and evaluate the derivative at the equilibria. What is the behavior (stability) of these equilibria?

c. How intense can the fishing be before this population of fish is driven to extinction? That is, find the value of h that makes the only equilibrium be zero.

11. (20pts) a. The population of the U. S. in 1900 was about 76.0 million, while in 1920, it was about 105.7 million. Assume that the population is growing according to the discrete Malthusian growth equation

$$P_{n+1} = (1 + r)P_n, \quad \text{with} \quad P_0 = 76.0,$$

where P_0 is the population in 1900 and n is in decades. Use the population in 1920 to find the value of r (to **4 significant figures**). **Note** that the population in 1920 is P_2, not P_1. Write the formula for the general solution to this model.

b. Estimate the population in 1960 based on this model. Given that the population in 1960 was 179.3 million, find the percent error between the actual and predicted values.

c. A Logistic growth model for the U. S. population in the 20^{th} century is given by

$$P_{n+1} = F(P_n) = 1.22P_n - 0.00049P_n^2,$$

where again n is in decades. If $P_0 = 76.0$, then use this model to predict the populations in 1910 and 1930.

d. Find the equilibria for this Logistic growth model. Calculate the derivative of $F(P)$ and evaluate it at the larger of the equilibria. What does this value say about the behavior of the solution near this equilibrium?