Basics about Fourier analysis

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PART ONE

Fourier analysis
On the menu ...

- Introduction - some history ...
- Notations.
- Fourier series.
- Continuous Fourier transform.
- Discrete Fourier transform.
- Properties.
- 2D extension.
An everyday challenge

Signal/image processing: need to:

- **analyze**:

- **synthesize**:

→ need some reference elements.
Biography

- Born in March 21th, 1768 at Auxerre (France), died in Mai 16th, 1830
- Graduated from ENS (Professors: Lagrange, Monge, Laplace)
- Chair at Polytechnique in 1797
- Elected member of the French Academy of sciences in 1817
- Elected member of the French Academy in 1826
# The Fourier revolution!

## Biography
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## Scientific contributions
- Analytic heat theory: modeling of heat propagation by trigonometric series (Fourier series)
- First to speak about the “greenhouse effect”
Notations

$L^1$ : space of integrable functions
Let $f \in L^1(\mathbb{R}^n)$ then $\|f\|_{L^1} = \int |f(t)| dt < \infty$

$L^2$ : space of functions of finite energy (square integrable)
Let $f \in L^2(\mathbb{R}^n)$ then $\|f\|_{L^2} = (\int |f(t)|^2 dt)^{\frac{1}{2}} < \infty$

Inner product between functions
Let $f, g \in E$ then $\langle f, g \rangle = \int f(t)\bar{g}(t) dt$

If $\exists T \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$, $f(t + T) = f(t)$ then $T$ is called the period of $f$ and $F = 1/T$ is the frequency of $f$.

Dirac function : $\delta(t)$
$\delta(t) = +\infty$ at $t = 0$, 0 otherwise and $\int \delta(t) dt = 1$.
Discrete case : Kronecker symbol : $\delta[n]$
$\delta[n] = 1$ if $n = 0$, 0 otherwise.
Fourier series: Definition

Idea: all periodic function of period $T$ can be decomposed as the sum of trigonometric polynomials $e^{j2\pi \frac{n}{T}t}$:

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n(f) e^{j2\pi \frac{n}{T}t}$$

where

$$c_n(f) = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) e^{-j2\pi \frac{n}{T}t}$$
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if $f$ is real

$$f(t) = a_0(f) + \sum_{1}^{+\infty} a_n(f) \cos \left(2\pi \frac{n}{T} t\right) + \sum_{1}^{+\infty} b_n(f) \sin \left(2\pi \frac{n}{T} t\right)$$
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where $a_0(f) = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) dt$, $b_0(f) = 0$ and for $n > 0$

$$a_n(f) = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \left(2\pi \frac{n}{T} t\right) dt, \quad b_n(f) = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \left(2\pi \frac{n}{T} t\right) dt$$
if $f$ is even then $c_{-n}(f) = c_n(f)$, if $f$ is real $b_n(f) = 0$,
if $f$ is odd then $c_{-n}(f) = -c_n(f)$, if $f$ is real $a_n(f) = 0$

Parseval equality :

$$\sum_{n=-\infty}^{+\infty} |c_n(f)|^2 = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \frac{1}{T} \int_0^T |f(t)|^2 dt = \|f\|_{L^2}^2.$$
The sinus/cosinus frequencies are multiple of $1/T$ (harmonics).
denote $e_n(t) = e^{j2\pi \frac{n}{T} t}$ then
denote \( e_n(t) = e^{j2\pi \frac{n}{T}t} \) then
\[
c_n(f) = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) \bar{e}_n(t) \, dt = \langle f, e_n \rangle
\]
Fourier series: Properties 3/3

- denote \( e_n(t) = e^{j2\pi \frac{n}{T} t} \) then
- \( c_n(f) = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) \bar{e}_n(t) dt = \langle f, e_n \rangle \)
- but \( \{e_n\} \) is an orthonormal basis (\( \langle e_n, e_m \rangle = 0 \) if \( n \neq m \) and 1 if \( n = m \))
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• but $\{e_n\}$ is an orthonormal basis ($\langle e_n, e_m \rangle = 0$ if $n \neq m$ and 1 if $n = m$)

$\implies$ Fourier series decomposition = projection a sinus/cosinus basis.
Real and odd signal with zero mean \( \Rightarrow a_n(f) = 0 \ \forall n \)
Fourier series: Example

Real and odd signal with zero mean
\[ \implies a_n(f) = 0 \ \forall n \]

Only \( b_n(f) \) are different from 0:

\[
\begin{align*}
    b_n(f) &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \left( 2\pi \frac{n}{T} t \right) dt \\
    &= \frac{2A}{T} \left[ -\int_{-T/2}^{0} \sin \left( 2\pi \frac{n}{T} t \right) dt + \int_{0}^{T/2} \sin \left( 2\pi \frac{n}{T} t \right) dt \right] \\
    &= \frac{2A}{T} \left\{ \left[ \frac{T}{2\pi n} \cos \left( 2\pi \frac{n}{T} t \right) \right]_{-T/2}^{0} + \left[ -\frac{T}{2\pi n} \cos \left( 2\pi \frac{n}{T} t \right) \right]_{0}^{T/2} \right\} \\
    &= \frac{A}{n\pi} (1 - \cos(n\pi) - \cos(n\pi) + 1) \\
    &= \frac{2A}{n\pi} (1 - (-1)^n)
\end{align*}
\]
Fourier series: Example

Basics about Fourier analysis
Continuous Fourier transform: Definition

Goal: generalization of the spectrum representation to non-periodic functions (frequencies $\nu \in \mathbb{R}$).

The Fourier transform of a function $f$ is given by

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi \nu t} dt$$

The inverse transform is given by

$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(\nu) e^{j2\pi \nu t} d\nu$$
## Continuous Fourier transform: Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Function</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( af_1(t) + bf_2(t) )</td>
<td>( a\hat{f}_1(\nu) + b\hat{f}_2(\nu) )</td>
</tr>
<tr>
<td>Dilation</td>
<td>( f(at) )</td>
<td>( \frac{1}{</td>
</tr>
<tr>
<td>Temporal Translation</td>
<td>( f(t + t_0) )</td>
<td>( \hat{f}(\nu)e^{j2\pi\nu t_0} )</td>
</tr>
<tr>
<td>Temporal Modulation</td>
<td>( f(t)e^{j2\pi\nu_0 t} )</td>
<td>( \hat{f}(\nu - \nu_0) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( f(t) \ast g(t) )</td>
<td>( \hat{f}(\nu)\hat{g}(\nu) )</td>
</tr>
<tr>
<td>Derivative</td>
<td>( f'(t) )</td>
<td>( j2\pi\nu \hat{f}(\nu) )</td>
</tr>
</tbody>
</table>

### Parseval-Plancherel theorem

The inner product is conserved:

\[
\int_{-\infty}^{+\infty} f(t) \bar{g}(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\nu) \bar{\hat{g}}(\nu) d\nu
\]

In particular:

\[
\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\nu)|^2 d\nu
\]
## Continuous Fourier transform: “Some classics”

<table>
<thead>
<tr>
<th>Function</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant $A$</td>
<td>$A\delta(\nu)$</td>
</tr>
<tr>
<td>Dirac $\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>Trigonometric function</td>
<td>$\cos(2\pi\nu_0 t)$</td>
</tr>
<tr>
<td>Sign function $\text{Sign}(t)$</td>
<td>$\frac{1}{j\pi\nu}$</td>
</tr>
<tr>
<td>Heavyside function $u(t)$</td>
<td>$\frac{1}{j\pi\nu} + \frac{1}{2}\delta(\nu)$</td>
</tr>
<tr>
<td>Square function $1$ if $-T/2 \leq t \leq T/2$, $0$ otherwise</td>
<td>$T\text{sinc}(\pi\nu T)$</td>
</tr>
<tr>
<td>Dirac comb $\sum_{m=-\infty}^{+\infty} \delta(t - mT_0)$</td>
<td>$\frac{1}{T_0} \sum_{n=-\infty}^{+\infty} \delta(\nu - n\nu_0)$</td>
</tr>
<tr>
<td>Gaussian $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$</td>
<td>$e^{-4\pi^2 \sigma^2 \nu^2}$</td>
</tr>
</tbody>
</table>
Discrete Fourier transform : Definition

Assume that $f(t)$ is sample on $N$ points at the frequency $F_e$, $f(nT_e)$ (we can directly note $f(n)$).

Discrete Fourier Transform (DFT) :

$$F(k) = \sum_{n=0}^{N-1} f(n)e^{-j2\pi \frac{nk}{N}}$$

Inverse transform :

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} F(k)e^{j2\pi \frac{nk}{N}}$$

Fast Algorithm : FFT
Saying that \( f(t) \) is sampled is equivalent to \( f(n) = f(t)P(t) \) where \( P(t) \) a Dirac comb associated to \( T_e \).

But the FT of the Dirac comb is a Dirac comb and a temporal product becomes a convolution product in the Fourier domain \( \implies \) duplication the input signal spectrum.
Shannon condition to have a correct reconstruction of the original signal: the support the FT of $f$ must be limited to the frequency range $[-F_e/2; F_e/2]$ in order to avoid some spectrum overlapping.
Truncation effect: $N$ samples $\Leftrightarrow$ to weight $f(t)$ by a square function.

$$f'(t) = f(t)\Pi(t) \quad \text{where} \quad \Pi(t) = 1 \quad \text{if} \quad t \in [0, NT_e], \ 0 \ \text{otherwise}$$

$\implies$ convolution in the spectral domain by a sinc function! We deform the spectrum!

Example: $f(t) = \sin(2\pi \nu t) \ (\nu = 30 \text{ Hz})$
Truncation effect: \( N \) samples \( \Leftrightarrow \) to weight \( f(t) \) by a square function.

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Example: \( f(t) = \sin(2\pi \nu t) \ (\nu = 30 \text{ Hz}) \)
To reduce the spectrum deformation, we can use other kind of windows $w(t)$ with “better” spectral behavior. Then $f'(t) = w(t)f(t)$ $\implies$ triangular, parabolic, Hanning, Hamming, Blackman-Harris, Gauss, Chebychev windows, ...
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Resolution power 1/2

When we use a square window, the principal sinc lobe in the Fourier domain has a total width of $2/N$ and the secondary lobes of $1/N$.

Problem when two pikes are too close: we can distinguish them or **resolve** them:
To resolve two frequencies, it is necessary that the gap between them is larger than the Fourier resolution:

\[ |\nu_1 - \nu_2| > \frac{1}{N} \]

Otherwise we can use the zero padding technique (virtual augmentation of \( N \) by inserting zeros between the signal’s samples, it is possible to prove that it does not alter the spectrum shape).
Localization behavior

- Signals well localized in time $\implies$ have large support in the frequency domain.
- Large signals in the time domain $\implies$ well localized in the frequency domain.

Ex: $\delta(t - t_0) \implies e^{j2\pi \nu t_0}$ (infinite support in the frequency domain)

Statistical information distribution

The quantities $\frac{|f(t)|^2}{E_f}$ and $\frac{|\hat{f}(\nu)|^2}{E_f}$ with $E_f$ the energy given by the Parseval theorem, can be interpreted probability densities of the information repartition in one domain or the other. We can compute the moments of these densities.
Time and frequency averages

\[
\bar{t} = \frac{1}{E_f} \int_{-\infty}^{+\infty} t |f(t)|^2 dt \quad \text{et} \quad \bar{\nu} = \frac{1}{E_f} \int_{-\infty}^{+\infty} \nu |\hat{f}(\nu)|^2 d\nu
\]

Time and frequency variances

\[
(\Delta t)^2 = \frac{1}{E_f} \int_{-\infty}^{+\infty} (t - \bar{t})^2 |f(t)|^2 dt \quad \text{et} \quad (\Delta \nu)^2 = \frac{1}{E_f} \int_{-\infty}^{+\infty} (\nu - \bar{\nu})^2 |\hat{f}(\nu)|^2 d\nu
\]

- \(\Delta \nu\) and \(\Delta t\) are invariants by translation in \(t\) and \(\nu\).
- The product \(\Delta t \Delta \nu\) is invariant time/frequency contraction/dilatation.
Gabor-Heisenberg incertitude principle

We can prove that:

$$\Delta t \Delta \nu \geq \frac{1}{4\pi}$$

Signals which are jointly of compact supports in both domains are gaussian signals.
All previously principles can be directly extended to the 2D case, for a function $f(x_1, x_2)$:

the continuous FT is given by:

$$\hat{f}(\nu_1, \nu_2) = \int_{-\infty}^{+\infty} f(x_1, x_2) e^{-j2\pi(\nu_1 x_1 + \nu_2 x_2)} dx_1 dx_2$$

and its inverse:

$$f(x_1, x_2) = \int_{-\infty}^{+\infty} \hat{f}(\nu_1, \nu_2) e^{+j2\pi(\nu_1 x_1 + \nu_2 x_2)} d\nu_1 d\nu_2$$
Images \( f(i, j) \) are assumed of size \( N \times M \)

**DFT:**

\[
F(k, l) = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} f(i, j) e^{-j2\pi \left( \frac{ki}{N} + \frac{lj}{M} \right)}
\]

**Inverse:**

\[
f(i, j) = \frac{1}{NM} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} F(k, l) e^{j2\pi \left( \frac{ki}{N} + \frac{lj}{M} \right)}
\]
2D extension: Lena

Basics about Fourier analysis
PART TWO

Time-frequency analysis
Next on the menu ...

- Time-frequency analysis
- Short Term Fourier Transform
- Limitations
Limitation of the Fourier transform

The FT has no “localization” notion: we cannot tell at which moment a frequency component appeared.

\[ \text{sinusoid (30 Hz) + Dirac (} t = 0.3\text{s)} \]
Chirp from 10Hz to 100Hz over 2s
Idea: “get a spectrum per instant $t$: time-frequency analysis”

But the FT is computed over $\mathbb{R}$: $\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi\nu t} dt \Rightarrow$ nonlocal.
We can get the “localization” by considering a “small” portion of the signal “close” to the considered instant.

⇒ signal windowing.
We can get the “localization” by considering a “small” portion of the signal “close” to the considered instant.

\[ \Rightarrow \text{signal windowing.} \]

The window is centered at \( \tau \) and we “slide” the window by tuning \( \tau \).

\[ f(t) \]

\[ w(t) \]
We get a 2D representation (time + frequency axis) called “the time-frequency plane” or spectrogram.
The Short Term Fourier Transform can be written

$$S_f(\nu, \tau) = \int_{-\infty}^{+\infty} w(t - \tau) f(t) e^{-j2\pi \nu t} dt$$

and we have

$$f(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S_f(\nu, \tau) w(t - \tau) e^{j2\pi \nu t} d\tau d\nu$$

where $w(t)$ can be one of the previous windows.
Important properties of the time-frequency plane

- Sampling grid in time and frequency

\[ \Delta t, \Delta \nu = \text{cst}. \]

\[ \tau \]

\[ \nu \]
Important properties of the time-frequency plane

- Sampling grid in time and frequency
- Gabor-Heisenberg (optimal case = gaussian window)
  \[ \Delta t \Delta \nu = \text{cst}. \]
Important properties of the time-frequency plane

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- \( \Delta t, \Delta \nu \) are fixed by \( w(t) \).
Important properties of the time-frequency plane

- Sampling grid in time and frequency
- Gabor-Heisenberg (optimal case = gaussian window) ⇒ $\Delta t \Delta \nu = \text{cst}$. 
- $\Delta t$, $\Delta \nu$ are fixed by $w(t)$. 
- ⇒ tiling of the time-frequency plane
Sinus + Dirac case

sinusoid (30 Hz) + Dirac
($t = 0.3s$)
Linear Chirp case

Chirp from 10Hz to 100Hz over 2s
Influence of the window size 1/2

L=32
Good time localization
Bad frequency localization
Influence of the window size 1/2

L=64
Fair time localization
Fair frequency localization
Influence of the window size $1/2$

$L = 128$
- Bad time localization
- Good frequency localization
Influence of the window size 2/2

Basics about Fourier analysis
Influence of the window size 2/2

Basics about Fourier analysis
Influence of the window size 2/2

Basics about Fourier analysis
Depending on the frequency content we can need several resolutions. For instance, let the following signal:

\[
f(t) = \begin{cases} 
2 \cos(2\pi 150t) & \text{if } t \in [0; 1s] \cup [0.4s; 0.78s] \cup [0.8s; 1s] \\
2 \cos(2\pi 150t) + 0.5 \cos(2\pi 30t) & \text{if } t \in [0.39s; 0.4s] \\
2 \cos(2\pi 150t) + \cos(2\pi 400t) & \text{if } t \in [0.78s; 0.8s] 
\end{cases}
\]
Limitations of the STFT 2/3

Basics about Fourier analysis
When different frequency component are present in the signal, it is better to have a STFT with a small window to analyze high frequencies and a wide window to analyze low frequencies. But this is impossible because the STFT provides a uniform time-frequency plane tiling!

⇒ Use of wavelets: multiresolution analysis.