Math 5C Practice Midterm

Lecture 2, S'03, Apr 24, 2003

All of your answers must be carefully justified. It is not enough to have a correct answer, you must explain how you got it. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. You may not use books, notes, and calculators on this exam.

Please, bring scratch paper. Pencil and eraser are recommended tools, and a ruler may come in handy. You may also have a $3'' \times 5''$ handwritten cheat sheet (both sides). The exam is based on the material covered through lecture on 4/24 (1.1-4 and 2.1-4 in the notes.)

1. (a) Find the Taylor series for $f(x) = \ln x$ at $x = 2$ by differentiating f a few times and observing a pattern.
   (b) What is the radius of convergence? What is the interval of convergence?
2. (a) Find the Taylor series for $\sinh x = (e^x - e^{-x})/2$ at $x = 0$. (Hint: you may use what you know about the Taylor series for e^x.)
   (b) For what x does the Taylor series for $\sinh x$ converge?
   (c) Recall $\cosh x = (e^x + e^{-x})/2$. Use $\frac{d}{dx} \cosh x = \sinh x$ and the result from part (a) to find the Taylor series for $\cosh x$ at $x = 0$. How do you know what the constant term is?
   (d) For what x does the Taylor series for $\cosh x$ converge?
3. Find the interval of convergence for the following power series.
   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n\,2^n}$.
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$.
   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$.
   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^2} (1-x)^n$.
   (e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n\, n!\, (x+5)^{2n-1}$.
4. Find the ordinary points and the singular points of the following DEs. For the singular points, say which are regular.
   (a) $(x^4 - x^2)y'' + (x^2 + x)y' + y = 0$.
   (b) $(t^2 - t - 2)y'' + (t+1)^2 y' + ty = 0$.
   (c) $x(1 - x^2)y'' + 3(x-1)y' - 7xy = 0$.
   (d) $t^2 y'' + \dfrac{2t}{1+t} y' + 2y = 0$.
   (e) $x^2 y'' + xy' + (x^2 - p^2)y = 0$
5. Assume that the solution of the DE

$$(2 - x^2)y'' + 2xy' - 2y = 0$$

   has a power series expansion about $x_0 = 0$.
   (a) Find the recursion formula.
   (b) Find two linearly independent solutions of the DE. What are the radii of convergence of these power series? Check your solutions by substituting them back into the DE.
   (c) Find the particular solution for the initial conditions $y(0) = 1$ and $y''(0) = 1$.

6. Use the method of Frobenius to find solutions of the DE

$$4xy'' + 2y' - y = 0.$$

(a) First notice that $x_0 = 0$ is a regular singularity. So assume $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Substitute this into the DE and determine the indicial equation and the recursion formula.
(b) Find two linearly independent generalized power series solutions. Check your solutions by substituting them back into the DE.
(c) What are the radii of convergence? (Hint: The Theorem of Frobenius will save you some computation.)
(d) (Extra credit) Can you figure out from the power series what these functions are? (BTW, this would give you an easy way to check your answer.)

7. Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{pmatrix}.$$

The Spectral Theorem tells you that there exists an orthogonal matrix B of determinant 1 such that $B^{-1}AB$ is diagonal.
(a) Find such a B.
(b) Solve the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

(c) Find a particular solution of the DE in part (b) when

$$\vec{x}(0) = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

8. Let

$$B = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

(a) Verify that B is an orthonormal set.
(b) Verify that B is linearly independent. Conclude that B is an orthonormal basis of $\mathbb{R}^3$. (Hint: Can you think of a clever way to use orthonormality to show that B is linearly independent?)
(c) Find the coordinates of

$$\vec{x} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

in terms of the basis B.

9. (a) Prove that if B is an orthogonal matrix, then $\det(B) = \pm 1$.
(b) Prove that if B is an orthogonal matrix, so is B^T.
(c) Prove that if B is an orthogonal matrix, so is B^n for any $n \geq 1$.
(d) Prove that if A is an invertible symmetric matrix, so is A^{-1}.
(e) Prove that if A is an invertible symmetric matrix, so is A^n for any $n \geq 1$.
(f) Let A_1 and A_2 be symmetric matrices. Can you conclude that $A_1 A_2$ is also symmetric?