Math 5C Practice Final

Lecture 2, S'03, Jun 3, 2003

The final exam is 7:30–10:30 on Tue (6/10) in Buchanan 1910.

Bring your student ID and scratch paper to the exam. Pencil and eraser are recommended tools. You may also have two $3'' \times 5''$ handwritten cheat sheets (both sides). You may not use books, notes, and calculators on this exam.

The final is comprehensive and is based on the material covered in lecture and sections 1.1–4, 2.1–4, 3.1–3, 4.1,2,4–6, 5.1–3 in the Lecture Notes. You may use the practice midterm as review exercises on Chapters 1 and 2. The exercises below cover Chapters 3–5.

All of your answers must be carefully justified. It is not enough to have a correct answer, you must explain how you got it. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers.

1. Define the following
   (a) The *Taylor series* of a function $f(x)$.
   (b) The *radius of convergence* of a power series.
   (c) The function $f(x)$ is *real analytic* at x_0.
   (d) *Ordinary*, *singular*, and *regular singular* points of the differential equation
   $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$
   (e) λ is an *eigenvalue* of the matrix A.
   (f) $\vec{v}$ is an *eigenvector* of the matrix A.
   (g) A is a *symmetric* matrix.
   (h) A is an *orthogonal* matrix.
   (i) A is a *diagonal* matrix.
   (j) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is *linearly independent*.
   (k) $\vec{w}$ is a *linear combination* of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.
   (l) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a *basis* of the vector space V.
   (m) $\vec{v}$ and $\vec{w}$ are *orthogonal* vectors. (They may not be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, so you can't just say the angle between them is $90°$. Hint: think inner product.)
   (n) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is *orthonormal*.
   (o) $f(x)$ is a *piecewise smooth* function on $\mathbb{R}$.
   (p) The *Fourier series* of a periodic function $f(x)$ whose period is T.
   (q) The *Fourier sine series* and *Fourier cosine series* of a function $f(x)$ on the finite interval $[0, L]$.
   (r) *Linear differential operator*.
   (s) λ is an *eigenvalue* of the linear differential operator $\mathcal{L}$.
   (t) f is an *eigenfunction* of the linear differential operator $\mathcal{L}$.
   (u) *Homogeneous* and *nonhomogenenous* boundary condition.
   (v) f is a *harmonic* function.

2. Find the Fourier series of the following periodic functions. Where the period is not stated, you should be able to figure it out from the definition of the function. State where you expect the function to be equal to its Fourier series. At points where values the function and its Fourier series are different, say what the value of the Fourier series is.
   (a) $f(x) = \sin^2(2x)$ with period $\pi/2$.

   (b) $f(x) = \begin{cases} 1 & \text{for } -\pi + 2k\pi \le x < 2k\pi \\ -1 & \text{for } 2k\pi \le x < \pi + 2k\pi \end{cases}$

   (c) $f(t) = \begin{cases} -1 - t & \text{for } -2 + 4k \le t < 4k \\ t - 1 & \text{for } 4k \le t < 2 + 4k \end{cases}$

   (d) $f(x) = \begin{cases} x & \text{for } -\pi \le x < \pi \\ f(x - 2k\pi) & \text{for } -\pi + 2k\pi \le x < \pi + 2k\pi \end{cases}$

3. Find the Fourier sine and cosine series of the following functions.
   (a) $f(x) = t - 1$ on $[0, 2]$.

   (b) $f(t) = t(1 - t)$ on $[0, 1]$.

   (c) $f(x) = e^x$ on $[0, \pi]$.

4. *Heat flow along a wire:* Fred Flintstone is sitting by a campfire roasting a 32-oz brontosaurus steak on a stick. As he feels the warm stick in his hand, it suddenly occurs to him: wouldn't it be nice to understand how the heat flows along it.
   Imagine you have a piece of wire of length L (in cm). The linear density of the wire at the point x is $\rho(x)$ (in g/cm), its heat capacity is $\sigma(x)$ (in $\frac{\text{J}}{\text{g C}^\circ}$) and its thermal conductivity is $\kappa(x)$ (in $\frac{\text{J cm}}{\text{s C}^\circ}$). Let $u(x, t)$ be the temperature of the wire at point x at time t. Derive the heat equation

$$\sigma(x)\,\rho(x)\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\,\frac{\partial u}{\partial x}\right)$$

   for the temperature of the wire.

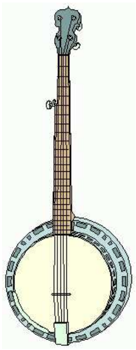
5. Find the solution of the initial value problem defined for $0 \le x \le 2$ and $0 \le t$

$$\begin{aligned} u_t(x, t) &= 4u_{xx}(x, t) \\ u(0, t) &= u(2, t) = 0 \\ u(x, 0) &= x - 1. \end{aligned}$$

6. We have seen that heat flow in a circular wire is described by the heat equation with a boundary condition that says $u(\theta, t)$ is periodic in θ. Find the solution of the initial value problem defined for $0 \le t$

$$\begin{aligned} u_t(\theta, t) &= 9u_{\theta\theta}(\theta, t) \\ u(\theta, t) &= u(\theta + 2\pi, t) \\ u(\theta, 0) &= |\theta - \pi|. \end{aligned}$$

7. *Vibrations of a string:*



(a) Show that the vibrations of an ideal string of length L (in cm) and linear density ρ (in g/cm) subject to tension T (in N), whose displacement at the point x at time t is $u(x,t)$ are described by the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}.$$

(b) Let $c^2 = T/\rho$ and $f(x)$ be any smooth function. Show that $u_1(x,t) = f(x-ct)$ and $u_2(x,t) = f(x+ct)$ are solutions of the above PDE. Can you explain what these solutions are in everyday terms?

8. Find the solution of the initial value problem defined for $0 \le x \le 5$ and $0 \le t$

$$\begin{aligned} u_{tt}(x,t) &= 25u_{xx}(x,t) \\ u(0,t) &= u(5,t) = 0 \\ u(x,0) &= 3\sin(2\pi x) - 5\sin(5\pi x) + 2\sin(8\pi x) \\ u_t(x,0) &= \sin(\pi x). \end{aligned}$$

9. *Heat flow in a 3-dimensional solid:* Wile E. Coyote has just purchased and ACME$^{\text{TM}}$Remote Bird Microwaving Device to roast Road Runner from a distance of a mile. Wile is a true connoisseur and he likes his roast roadrunner slightly pink, but not raw. In order to achieve the perfect result, he needs to understand how the heat transmitted by the device flows in a 3-D object, such as Road Runner.
Let $u(x,y,z,t)$ be the temperature at the point (in C°) (x,y,z) at time t, $\rho(x,y,z)$ the density (in g/cm^3), $\sigma(x,y,z)$ the heat capacity (in $\frac{\text{J}}{\text{g C}^\circ}$), and $\kappa(x,y,z)$ the thermal conductivity (in $\frac{\text{J}}{\text{cm s C}^\circ}$) of a 3-dimensional ideal solid. Show that the temperature of this solid is given by the PDE

$$\sigma(x,y,z)\,\rho(x,y,z)\,\frac{\partial u}{\partial t}(x,y,z,t) = \nabla\big(\kappa(x,y,z)\,\nabla u(x,y,z,t)\big).$$

10. Find the solution of the initial value problem defined for the 2-D region $0 \le x \le 2$, $0 \le y \le 3$ and $0 \le t$

$$\begin{aligned} u_t(x,y,t) &= 4\big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\big) \\ u(0,y,t) &= u(2,y,t) = u(x,0,t) = u(x,3,t) = 0 \\ u(x,y,0) &= (x-1)\big(3\sin(\pi y) - 4\sin(4\pi y)\big). \end{aligned}$$