Bessel's Equation

Three equivalent forms of Bessel's equation:

1. $x\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) + (x^2 - p^2)y = 0$

2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - p^2)y = 0$

3. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} + \dfrac{x^2 - p^2}{x^2}y = 0$

We will use the 2nd form. Notice that $x = 0$ is a regular singular point. Hence we may assume there is a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Hence

$$\begin{aligned}
x^2\frac{d^2y}{dx^2} &= x^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} \\
&= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} \\
x\frac{dy}{dx} &= x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \\
&= \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} \\
(x^2-p^2)y &= x^2y-p^2y \\
&= x^2\sum_{n=0}^{\infty} a_nx^{n+r} - p^2\sum_{n=0}^{\infty} a_nx^{n+r} \\
&= \sum_{n=0}^{\infty} a_nx^{n+r+2} - \sum_{n=0}^{\infty} p^2a_nx^{n+r} \\
&= \sum_{n=2}^{\infty} a_{n-2}x^{n+r} - \sum_{n=0}^{\infty} p^2a_nx^{n+r}
\end{aligned}$$

Substitute these into the DE:

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \\
&\quad \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \\
&\quad \sum_{n=2}^{\infty} a_{n-2}x^{n+r} - \sum_{n=0}^{\infty} p^2 a_n x^{n+r} \\
&= (r^2 - p^2)a_0 x^r + \\
&\quad \Big((r+1)^2 - p^2\Big)a_1 x^{r+1} + \\
&\quad \sum_{n=2}^{\infty} \Big[\Big((n+r)^2 - p^2\Big)a_n + a_{n-2}\Big] x^{n+r}
\end{aligned}$$

The lowest power of x is x^r, so the indicial equation is

$$r^2 - p^2 = 0 \;\Rightarrow\; r = \pm p,$$

and the recursion formula is

$$a_n = \frac{-a_{n-2}}{(n+r)^2 - p^2} = \frac{-a_{n-2}}{(n+r+p)(n+r-p)}.$$

Since $r = \pm p$, $(r+1)^2 - p^2 \neq 0$, hence $a_1 = 0$, and then $a_n = 0$ for odd n.

We will find one nonzero solution. Let's choose $r = p$. When $n = 2k$,

$$\begin{aligned} a_{2k} &= \frac{-a_{2k-2}}{(2k+2p)(2k)} = \frac{-a_{2k-2}}{4(k+p)k} \\ &= \frac{a_{2k-4}}{4^2(k+p)(k-1+p)k(k-1)} \\ &\;\;\vdots \\ &= \frac{(-1)^k a_0}{4^k(k+p)(k+p-1)\cdots(p+1)k!} \\ &= \frac{p!\,(-1)^k\,a_0}{4^k(k+p)!k!} \end{aligned}$$

Hence

$$\begin{aligned} y &= p!\,a_0\,x^p \sum_{k=0}^{\infty} \frac{(-1)^k\,x^{2k}}{2^{2k}(k+p)!k!} \\ &= (2^p\,p!\,a_0)\left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+p)!k!}\left(\frac{x}{2}\right)^{2k} \end{aligned}$$

The Bessel equation is linear and homogeneous, so any multiple of the above solution is still a solution. Hence we can conveniently choose $(2^p\,p!\,a_0) = 1$.

The function

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+p)!k!}\left(\frac{x}{2}\right)^{2k}$$

is called the *p-th Bessel function of the first kind*.

The other linearly independent solution is obtained by using $J_p(x)$ and variation of parameters. It's called the *p-th Bessel function of the second kind*.