Math 579 Exam 2 Solutions

Oct 24, 2007

1. (10 pts) For $n \in \mathbb{Z}^+$, let $\phi(n)$ be the number of positive integeres $m \leq n$ that are relatively prime to n. Recall that we proved

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$$

where $p_1, p_2, \ldots, p_r$ are the distinct prime factors of n.

Prove that if m and n are relatively prime positive integers, then $\phi(mn) = \phi(m)\phi(n)$. (Hint: if m and n are relatively prime, what can you say about their prime factors?)

Consider the standard prime factorizations $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ (where the p_i are all distinct) and $n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}$ (where the q_i are all distinct). Since m and n are relatively prime, $p_i \neq q_j$ for all i, j. Hence the distinct prime factors of

$$mn = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}$$

are exactly $p_1, \ldots, p_r, q_1, \ldots q_s$. Therefore

$$\begin{aligned}
\phi(mn) &= mn \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right) \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_s}\right) \\
&= m \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \; n \prod_{i=1}^{s} \left(1 - \frac{1}{q_i}\right) \\
&= \phi(m)\phi(n)
\end{aligned}$$

2. (15 pts) A catering company brings fifty identical hamburgers to a party with twenty guests.
   (a) How many ways can the hamburgers be divided among the guests?

   We proved in class that the number of ways to divide n identical objects into k lots (where a lot is allowed to be empty) is $\binom{n+k-1}{k-1}$. (We actually gave two proofs, one with generating functions, and one I called the picket fence argument.) So the answer is $\binom{69}{19}$. The picket fence argument works very nicely to show this. The caterer could divide the burgers by the following method. She starts by lining up 69 plates. Then she picks 19 of those plates and replaces them with cocktail napkins. This divides the remaining 50 plates into 20 groups with the napkins serving as the dividers. If two napkins are adjacent then the group between them consists of 0 plates. Now she just has to put burgers on the plates and hand them out to the guests.

   (b) How many ways can the hamburgers be divided among the guests if every guest must receive at least one hamburger?

   This is very similar to the previous problem. First we give one burger to each guest. Now the question is how many ways there are to divide the remaining 30 burgers among the 20 guests. By the same logic as in part (a), the number of ways to divide the remaining 30 burgers among 20 people is $\binom{30+20-1}{20-1} = \binom{49}{19}$.

3. (15 pts) Use an appropriate generating function to answer the following question. A Halloween candy bowl has three different types of candy bars: 3 Musketeers, Almond Joy, and Reese's Peanut Butter Cups. There are many candy bars of each type. How many different ways are there to pick 6 candy bars from the bowl (if the order doesn't matter)?

First, let x_1, x_2, x_3 represent the three types of candy bars respectively. The generating function

$$f(x_1, x_2, x_3) = (x_1^0 + x_1^1 + x_1^2 + \cdots)(x_2^0 + x_2^1 + x_2^2 + \cdots)(x_3^0 + x_3^1 + x_3^2 + \cdots)$$

generates all the possible outcomes of picking some candy bars out of the bowl (assuming that there are an unlimited number of each type). E.g. $x_1^2 x_2^3 x_3^1$ would represent getting 2 3 Muskateers, 3 Almond Joys, and 1 peanut butter cup. So the number we are looking for is the number of terms of total degree 6. A good way to count these is to let $x_1 = x_2 = x_3 = x$ and find the coefficient of x^6 in the resulting function

$$f(x) = (1 + x + x^2 + \cdots)^3.$$

Using the usual power series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

we get $f(x) = (1-x)^{-3}$. Let the Maclaurin series expansion of f be

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n.$$

The usual formula from calculus says

$$\alpha_6 = \frac{f^{(6)}(0)}{6!}.$$

To compute this, notice that

$$\frac{d}{dx}(1-x)^{-n} = n(1-x)^{-n-1}$$

so

$$\frac{d^6}{dx^6}(1-x)^{-3} = 3 \cdot 4 \cdots 8(1-x)^{-9}.$$

Hence

$$\alpha_6 = \frac{3 \cdot 4 \cdots 8}{6!} = 28.$$

So there are 28 ways to pick 6 candy bars from the bowl.

4. (10 pts) Again, let the ϕ function be defined as in problem 1. Find all numbers $n \in \mathbb{Z}^+$ such that $\phi(n) = 7$ or if there is no such number, show why not.

Let $n \in \mathbb{Z}^+$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be its standard prime factorization. Now

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}{p_1 p_2 \cdots p_r} \prod_{i=1}^{r} (p_i - 1)$$

$$= p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_r^{\alpha_r - 1} \prod_{i=1}^{r} (p_i - 1).$$

How can this be equal to 7? The only way is if one of the terms is 7 and the rest are 1. But $p_i - 1$ cannot be 7 because this would make $p_i = 8$, which is not a prime. So there must be some i such that $p_i^{\alpha_i - 1} = 7$. The only way this can happen is if $p_i = 7$ and $\alpha_i = 2$. But then $p_i - 1 = 6$ and

$$p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_r^{\alpha_r - 1} \prod_{i=1}^{r} (p_i - 1) \geq 56$$

So whatever we do, $\phi(n)$ cannot be 7. Hence there exists no $n \in \mathbb{Z}^+$ such that $\phi(n) = 7$.

5. (10 pts) **Extra credit problem.** In Mandarin Chinese culture, 6 and 8 are considered lucky numbers. A raffle ticket has a six-digit number on it. If the number has at least a 6 and at least an 8, it is considered especially lucky. (For example, the number 37<u>6</u>9<u>8</u>0 is especially lucky.) How many such raffle tickets are especially lucky? (Hint: use the Principle of Inclusion and Exclusion.)

   Let S_1 be the set of all those 6-digit numbers that do not contain 6 and the S_2 the set of all those 6-digit numbers that do not contain 8. Note that the numbers in S_1 consist of the digits $0, 1, 2, 3, 4, 5, 7, 8, 9$. Hence there are 9^6 of them. For the same reason, S_2 also has 9^6 elements. Notice that $S_1 \cap S_2$ is the set of 6-digit numbers that contain neither 6 nor 8. So they consist of the digits $0, 1, 2, 3, 4, 5, 7, 9$. Therefore $|S_1 \cap S_2| = 8^6$. Using the Principle of Inclusion and Exclusion, we get

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| = 2(9^6) - 8^6$$

   Observe that $S_1 \cup S_2$ contains all those 6-digit numbers that are not especially lucky. Hence there are $10^6 - 2(9^6) + 8^6 = 199262$ 6-digit numbers that are especially lucky. In case you care to know, this is about 20% of the tickets.