Math 521A Exam 2 Solutions

Oct 17, 2007

1. (10 pts) Define the relation R on $\mathbb{Z}$ by aRb if $a+b$ is even.
   (a) Prove that R is an equivalence relation.

   Let $a, b, c \in \mathbb{Z}$.

   Reflexivity: Since $a + a = 2a$ is clearly even, aRa.

   Symmetry: Suppose aRb. Then $a+b$ is even. Hence $b+a = a+b$ is also even. This shows bRa.

   Transitivity: Suppose aRb and bRc. Then $a+b$ and $b+c$ are even. This means there exist $x, y \in \mathbb{Z}$ such that $a+b = 2x$ and $b+c = 2y$. Now

   $$a + c = (a+b) + (b+c) - 2b = 2x + 2y - 2b = 2(x+y-b)$$

   which is obviously even. This shows aRc.

   (b) What are the equivalence classes of R?

   Notice that

   $$[0] = \{x \in \mathbb{Z} | 0 + x \text{ is even}\} = \{x \in \mathbb{Z} | x \text{ is even}\}$$

   and

   $$[1] = \{x \in \mathbb{Z} | 1 + x \text{ is even}\} = \{x \in \mathbb{Z} | x \text{ is odd}\}.$$

   Notice that every integer must be in one of these. So we have already found all the equivalence classes. There are only two: the set of even numbers and the set of odd numbers. Of course we could have chosen any even number as a representative of the first and any odd number as a representative of the second.

2. (10 pts) Let p be a prime and $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. Prove that if $p | a_1 a_2 \cdots a_n$, then $p | a_i$ for some $1 \le i \le n$.

   We will do this by induction. If $n = 1$ then the statement is that if $p|a_1$ then $p|a_1$. This is obviously true.

   Assume that the statement is true for some positive integer n. We will show it is also true for $n+1$. Suppose $p | a_1 a_2 \cdots a_{n+1}$. Then $p | (a_1 a_2 \cdots a_n) a_{n+1}$. By Euclid's Lemma, $p | (a_1 a_2 \cdots a_n)$ or $p | a_{n+1}$. If $p | (a_1 a_2 \cdots a_n)$ then by the inductive hypothesis, $p | a_i$ for some $1 \le i \le n$. Otherwise $p | a_{n+1}$. So $p | a_i$ for some $1 \le i \le n+1$.

3. (10 pts) Find a relation $\sim$ on a set S which is symmetric and transitive, but not reflexive. Explain why your relation is symmetric and transitive, but not reflexive.

   Let S be any nonempty set and $\sim$ such that $a \sim b$ is never true. This is called the empty relation. (Indeed, if you think of $\sim$ as a subset of $S \times S$, then $\sim = \emptyset$.) Now take any element $a \in S$ and observe that $a \not\sim a$. Hence $\sim$ is not reflexive. The reason $\sim$ is symmetric is that whatever $a, b \in S$ might be, in the statement "if $a \sim b$ then $b \sim a$" the condition is always false, hence the statement is true. Transitivity holds for the same reason.

   If you would like a more interesting example of such a relation, try $S = \mathbb{Z}$ and $a \sim b$ if $ab \neq 0$. Why does this fail to be reflexive? Why is it symmetric and transitive?

4. (10 pts)

(a) State the definition of a partition of a set S.

A partition $\mathcal{P}$ of S is a set of nonempty subsets of S such that
(i) they are pairwise disjoint,
(ii) their union is S.

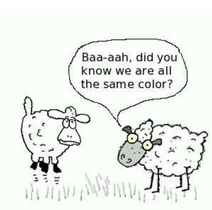
(b) Let S be a nonempty set and $\mathcal{P}$ a partition of S. Show that there exists an equivalence relation $\sim$ on S whose equivalence classes are the sets in $\mathcal{P}$.

How to define such a relation is on p. 20 in your text. We did the rest of the proof in class. It should be in your lecture notes.

5. (10 pts) Prove that if $n \in \mathbb{Z}$ and $n \geq 2$ then n can be expressed as $n = p_1 p_2 \cdots p_k$ where p_i is a prime for all $1 \leq i \leq k$. (Hint: strong induction.)

See p. 16 in the text.

6. (10 pts) **Extra credit problem.**
Find the mistake in the following argument and explain why it's a mistake.



The following is a proof that there exist no black sheep. First, we will prove that in any group of sheep, every sheep has the same color by doing induction on the number of sheep in the group.

It is obvious that in any group of sheep which consists of exactly one sheep, every sheep has the same color. This establishes the base case.

The inductive hypothesis is that in every group of n sheep, every sheep has the same color.

Now look at a group of $n+1$ sheep. Let's pick one, set it aside, and look at the rest of the animals. They form a group of n sheep, therefore they all have the same color by the inductive hypothesis. Now we will prove that the sheep we set aside has the same color too. Let's pick another sheep and switch it with the sheep we set aside. We still have a group of n sheep, therefore they all have the same color by the inductive hypothesis. Hence the sheep we first set aside has the same color as all the others.

The above argument shows that every sheep on earth has the same color. I suppose that you've seen a white sheep before. Now you know that every other sheep must also be white. Hence there exist no black sheep despite any rumor you might have heard to the contrary.

No, the mistake is not that the result contradicts reality. The fact that there indeed exist both black and white sheep–see the herds grazing along Highway 111 just north of Calexico–only tells you that the argument must have a mistake in it, but is not itself the mistake.

Since it is obviously false that in any set of sheep all the animals are the same color, there must be an error in the inductive argument. It is not that I didn't prove the inductive hypothesis. As its name suggests, the inductive hypothesis is only meant to be assumed, not proved.

You can find the mistake by trying how the induction supposedly goes from the base case to $n = 2$. Of course, before you go looking for a mistake, make sure you understand why the argument seems to work in going from n sheep to $n+1$ sheep.