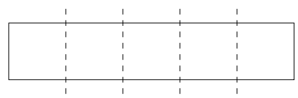
Math 413 Final Exam Solutions

Dec 11, 2007

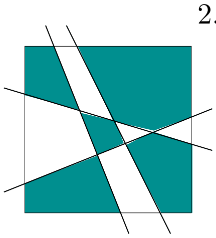
1. (10 pts) How many Cuisenaire trains of length n can you build out of Cuisenaire rods? Assume that you distinguish between the front and the back of the train. Make sure you fully justify your answer.

Think about the problem backwards (reversibility!). Instead of combining shorter rods to make a rod of length n, start with a rod of length n and cut it up into shorter rods. Since the length is n, there are $n-1$ places to cut. (Just imagine that if you did all $n-1$ cuts, you'd cut the rod into exactly n pieces.) Here is the example of a train of length 5 with 4 potential places to cut.

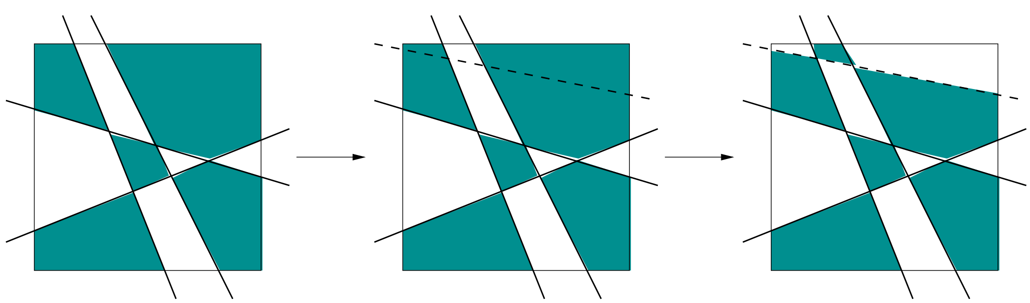


So you make a choice at every one of the $n-1$ places where you can cut whether to cut or not. This means you are choosing between 2 options. The choices are independent of each other, so there are 2^{n-1} different ways to make all $n-1$ choices. We get every possible train this way and each choice results in a different train.

2. (10 pts) Take a square and draw several straight lines across it. This divides the square into several regions. We will call two regions adjacent if they share a line segment. Two regions that only have one point in common are not considered adjacent. Give a compelling mathematical argument that at most two colors are always sufficient to color all the regions in such a way that no two adjacent regions have the same color.



For simplicity, let's say the two colors are black and white. Let's draw the lines one by one and always color the square with two lines as we go. Notice that the statement is true when you draw no lines at all (actually one color is sufficient in this case). Now do the following. Every time you draw a new line, it will cut through some regions, dividing them into two parts. These two adjacent parts will now have the same color. We will fix this problem by choosing one side of the line and reversing the colors of all the regions on that side of the line. Notice that this fixes the problem we have had: two newly adjacent regions on the two sides of the line will now have different colors. Here is an illustration of this with the last line drawn as a broken line:



Reversing the colors does not introduce any new color conflicts either, as any regions that were adjacent prior to drawing the line, and hence had different colors, will still have different colors, only those colors will now be switched. This shows that no matter how many lines we draw, we can always color the resulting regions with at most two colors.

3. (10 pts) Find all the ways in which 45 can be written as a sum of two or more consecutive positive integers. Explain how you found these sums and how you know that these are all the possibilities.

Here is one way to look at this problem. Suppose you some consecutive integers $x_1, x_2, \dots, x_n$. Notice that if n is odd, then there is a number in the middle of this list. For example, in $5, 6, 7, 8, 9$, the number 7 is in the middle. Call this number x. Then you can rewrite your list as

$$x-k, x-(k-1), \dots, x-1, x, x+1, \dots, x+(k-1), x+k.$$

Actually, we can even tell $n = 2k+1$, which is why we needed n to be odd. Now when you add these numbers, you get

$$x-k+(x-(k-1))+\cdots+x+\cdots+(x+(k-1))+(x+k) = (2k+1)x$$

as the $-k$ at the beginning and the $+k$ at the end cancel, etc.

If n is instead even, then there is no middle number, but you can still do something similar. Notice that in this case, you can just take pairs of numbers from the beginning and the end of the list and their sum is always going to be constant. Namely

$$x_1 + x_n = x_2 + x_{n-1} = x_3 + x_{n-2} = \cdots = x_{n/2} + x_{n/2+1}.$$

There are of course $n/2$ such pairs of numbers, so you get

$$x_1 + x_2 + \cdots + x_n = \frac{n}{2}(x_1 + x_n) = \frac{n}{2}(x_{n/2} + x_{n/2+1}).$$

But $x_{n/2}$ and $x_{n/2+1}$ are consecutive numbers, so $x_{n/2+1} = x_{n/2} + 1$ and $x_{n/2} + x_{n/2+1} = 2x_{n/2} + 1$. This is an odd number (it's two times an integer plus 1). So

$$x_1 + x_2 + \cdots + x_n = \frac{n}{2}(2x_{n/2} + 1).$$

Notice that since n was even, $n/2$ is an integer.

In either case, we get that the sum of the consecutive numbers is some (positive) odd number times some other integer. So the key to finding all possible ways to write 45 as a sum of consecutive integers is to look for ways to write 45 as a product of an odd number with another integer. Since 45 is odd, all its divisors are odd. Here are all the ways to write 45 as a product of two positive integers:

$$45 = 1 \cdot 45 = 3 \cdot 15 = 5 \cdot 9.$$

This shows that the following cases may be possible.

- 1 consecutive number, the middle one is 45. This is of no interest, as we want to write 45 as a sum of two or more consecutive integers.
- 3 consecutive numbers, the middle one is 15. This gives $45 = 14 + 15 + 16$.
- 5 consecutive numbers, the middle one is 9. This gives $45 = 7 + 8 + 9 + 10 + 11$.
- 9 consecutive numbers, the middle one is 5. This gives $45 = 1+2+3+4+5+6+7+8+9$.
- 15 consecutive numbers, the middle one is 3. Then the first number would have to be $3 - 7 = -4$ (because $15 = 2 \cdot 7 + 1$), which is not positive. So this does not give a result of interest to us.
- 45 consecutive numbers, the middle one is 1. This does not give a result of interest for the same reason as the previous case.
- 2 consecutive numbers, the middle two are 22 and 23 (because $45 = 2 \cdot 22 + 1$). This gives $45 = 22 + 23$.
- 6 consecutive numbers, the middle two are 7 and 8 (because $15 = 2 \cdot 7 + 1$). This gives $45 = 5 + 6 + 7 + 8 + 9 + 10$.
- 10 consecutive numbers, the middle two are 4 and 5 (because $9 = 2 \cdot 4 + 1$). This gives $45 = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$, but 0 is not a positive integer.

- 18, 30, or 90 consecutive numbers. In light of the previous case, these would clearly have to involve some negative integers. So they are of no interest to us.

So the 5 ways to write 45 as a sum of two or more consecutive positive integers are

$$45 = 22 + 23 = 14 + 15 + 16 = 7 + 8 + 9 + 10 + 11$$
$$= 5 + 6 + 7 + 8 + 9 + 10 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9.$$

Many of you gave correct solutions involving some intelligent trial and error. This is fine as it can correctly solve this problem. But notice that the above method has the advantage that it readily generalizes to any other number and provides an effective (of somewhat tedious) method of finding all the ways to write a positive integer as a sum of two or more consecutive positive integers.

Historical note: the idea at the crux of this approach (adding numbers from the beginning and end of the list) is usually attributed to Gauss. He was about 12 years old at the time. This was in the 18th century.

4. (10 pts) Santa Claus is doing his Christmas shopping at the North Pole Toy Factory. Santa has a loyalty card with this store which gives him a general 10% discount. He also has a coupon he has clipped from the ad section of the Arctic Daily. The coupon is worth an additional 15% of the entire bill. There is currently no sales tax at the North Pole. As Santa gets to the cashier, he presents both his loyalty card and the coupon. The crafty elf at the register says "So which of these would you like me to scan first?"

(a) What should Santa choose? Does it make a difference?

Let the original cost of Santa's purchase be x. Suppose the elf scans the coupon and then the loyalty card. The price is first reduced to $0.85x$, then to $0.9(0.85x) = 0.765x$. Suppose the elf scans the loyalty card and then the coupon. The price is first reduced to $0.9x$, then to $0.85(0.9x) = 0.765x$.
As you can see, Santa obtains the same 23.5% combined discount in both cases. It does not matter in what order his discounts are applied.

(b) What basic principle of algebra is at the heart of your answer to the previous question?

It is actually two properties of multiplication, associativity and commutativity, that explain this result.

$$\begin{aligned} 0.9(0.85x) &= (0.9 \cdot 0.85)x && \text{by associativity} \\ &= (0.85 \cdot 0.9)x && \text{by commutativity} \\ &= 0.85(0.9x) && \text{by associativity.} \end{aligned}$$

5. (10 pts) You read about Dr. Polya's four phases of problem solving this semester.

(a) What are the four phases according to Dr. Polya? Give a brief description of what you would do in each phase.

Understanding: Understand the problem, familiarize yourself with the math, try some examples, visualize the problem.

Make a plan: Design a plan for solving the problem. Think about similar problems you have solved in the past. Draw from past experience.

Carry out the plan: Follow the plan you made in the previous step. Think about whether your partial answers make sense and check each step as you go.

Look back: Read your solution again. Does it answer the question? Is the answer reasonable? Check your solution. Can you follow your own steps? Is there a better way to do the problem? Can you generalize your solution? What have your learned from solving this problem that you can potentially use in the future?

(b) What are the advantages of such a four-phase approach in solving math problems?

The four phases break the problem down into more manageable pieces. This approach helps organize your thoughts and produce an answer that someone else can follow. Going through the last phase makes you check your answer one more time and gain problem solving experience by reflecting on what you did and why to solve the problem.

6. (10 pts)

(a) List the three main traits of algebraic thinking. Give an example for each from one of the exercises on this exam, or if a trait does not appear on this exam, say so. (Your examples can come from several different exercises.)

Abstraction, generalization: One example is how re-coloring the square after drawing each line gives a general proof that the square can be colored with two colors after drawing any number of lines through it. Finding commutativity and associativity in problem 4.(b) is an example of abstraction.

Patterns, functions: In problem 1, we came up with a function giving the number of trains that can be build in terms on the length of the train.

Reversibility: We solved problem 1 by reversing the building of the train and thinking about cutting up a rod of length n instead.

(b) What role do specialization and generalization play in mathematical problem solving? At what stage of solving the problem are you likely to use each?

Specialization lets you think about the problem in concrete terms and develop a sense for what math you need to solve it. You are likely to do this at the beginning stage. Generalization lets you take a step away from the specifics of the problem and come up with a method which can be adapted to solving similar problems. It also let's you develop a deeper understanding of the math you use to solve the problem. You are likely to do this at a later stage, for example when you look back on the problem.

7. **The pirates do not give up.** As the pirating business experiences a boom, the captain hires a new pirate. So there are now 6 of them. Consider the following two scenarios.

(a) (6 pts) The six pirates raid a ship and steal 100 gold coins. They now have to divide up the loot. The pirates are all extremely intelligent, mean, and selfish. As usual, they agree that the captain will propose a distribution of the loot. All pirates (including the captain) vote on the proposal, and if half the crew or more go "Aye", the loot is divided as proposed. No pirate would be willing to take on the captain without superior force on their side.

If more than half the crew–which includes the captain–vote against the captain's proposal, there is a mutiny, and all pirates turn against the captain and kill him. Then the pirates elect the most senior pirate captain and start over again with distributing the loot.

What is the maximum number of coins the captain can keep without risking his life?

Let's label the pirates P1 through P6 in increasing order of seniority. We will now analyze what happens if some of the more senior pirates are removed from the picture. We will label the cases with the pirates that are still alive:

P1, P2: P2 would have no reason to give anything to P1, as P1 can't vote P2's proposal down by himself.

P1, P2, P3: If P3 is killed, P1 gets none of the loot. So P3 can offer just one coin to P1, and P1 will certainly vote for his proposal. P3 does not have to offer any money to P2, as he already has the two votes he needs. In fact, no matter how much money he would offer to P2, P2 will likely vote against him, as he would get all 100 coins if P3 were dead. If P3 were to offer no money to P1, then he would risk being voted down by both P1 and P2. Since P3 is intelligent, he will not risk this.

P1, P2, P3, P4: If P4 were killed, P2 would get none of the loot (see previous case). So P4 can offer just one coin to P2 and keep the rest to himself. This will be enough to buy P2's good will. Trying to keep all the money to himself is a risky strategy for P4, just like in the previous case, so he will not do this.

P1, P2, P3, P4, P5: P5 needs three votes for his proposal to survive. He can give just one coin each to P1 and P3. Since P1 and P3 would get no money from P4 if P5 were killed, this is enough to buy their votes. This way P5 can keep 98 coins. Trying to keep any more than that, he would run the risk of being killed, as he would no assurance of having two friends among the pirates. Notice this is the answer to the extra credit question on the first exam.

P1, P2, P3, P4, P5, P6: P6 needs two votes on his side. Following the same logic as before, he can offer a coin each to P2 and P4 to buy their votes. Since P2 and P4 would get nothing from P5 if P6 were killed, they will vote for P6's proposal if it awards them any money at all. So P6 can keep 98 coins to himself. Trying to keep any more runs the risk of not having enough pirates on his side.

(b) (9 pts) The six pirates raid a ship and steal only 1 gold coin. They set out to divide up the loot according to the same rules as in part (a). Only now they are angry that they all risked their lives for so little gain, and in their anger, they thirst for blood. Each pirate's priorities are in order:

(i) his life

(ii) money

(iii) killing fellow pirates (so next time, the loot has to be divided among fewer people).

That is if a pirate has a choice between two outcomes in which he gets the same amount of money, he will choose the one in which more of his fellow pirates are killed.

What distribution should the captain propose if he wants to survive?

We can use the same strategy as in part (a) and analyze the cases.

P1, P2: P2 could keep the coin to himself and P1 could do nothing about it.

P1, P2, P3: P3 needs one more friend on his side. If P3 were dead, P2 could keep the coin. So he will vote against P3 no matter what. If P3 does not offer the coin to P1, then P1 will vote for killing him. He still won't get money if P3 is dead, but he voting for killing P3 would satisfy his 3rd priority. So the only thing P3 can do to survive is to offer the coin to P1. In that case, P1 will vote for P3, because if P3 were dead, P2 would keep the coin to himself.

P1, P2, P3, P4: P4 needs one more friend. If he tried to keep the coin to himself, everyone would vote against him, as they are all happier with him dead in that case. So he has to give it away. He could offer it to P3 or P2, as neither of them would get money if P4 were dead.

P1, P2, P3, P4, P5: P5 needs two more friends. This is really problematic, as he has only one coin to give away. He could use it to buy the vote of any of the other pirates. P1 and P4 would certainly be happy with a coin, as neither would get money if P5 were dead. Even P2 and P3 would be happy with a coin, as neither could be sure that he would get the coin from P4 if P5 were killed. But whichever pirate gets the coin, the other three are going to vote against P5, as they have nothing to lose with his death, and they can satisfy their anger that way. So P5 can do nothing to assure his survival. This is the answer the extra credit problem on the second exam.

P1, P2, P3, P4, P5, P6: P6 needs three friends. The lucky thing for P6 is that he can always count on P5 to vote for his proposal. P5 really does not want P6 dead, because then it would be his turn to be captain, and that would be a very bad situation for him as we saw in the previous case. If P6 dies, so does P5.
P6 can use the coin to buy another vote. He could actually offer the coin to any of P1, P2, P3, or P4. As we explained in the previous case, any of these pirates should be happy with a firm offer of a gold coin, as they risk getting nothing if P6 and P5 were dead. So the survival strategy for P6 is to offer the coin to P1, P2, P3, or P4 and count on P5 voting for him as well.