Math 3C Midterm Solutions

White version of exam

Lecture 2, W'03, Feb 12, 2003

1. (2 pts for each correct answer, -1 pt for each wrong answer, 0 for blank) Decide if the following statements are true (i.e. always true) or false (i.e. false in at least one case). You need not justify your answer.
   (a) [T] Every autonomous first-order DE is separable.
   It looks like $y' = f(y)$, so the RHS is clearly a product of a function of x and y.
   (b) [F] The DE $e^{x^2}y' + e^{x+y} = 0$ is linear.
   The e^y part violates linearity: $e^{y_1+y_2} \neq e^{y_1} + e^{y_2}$.
   (c) [T] The DE $x^2y'' + \frac{y'}{\cos(x)} = 0$ is a linear and homogeneous.
   Verify that the LHS satisfies both properties of a linear operator. and note that the RHS is 0.
   (d) [T] If L is a linear operator and y_1 and y_2 are differentiable functions, then $L(y_1) + L(y_2) = L(y_1 + y_2)$.
   By definition of linearity.
   (e) [F] If L is an operator and $L(y_1 + y_2) = L(y_1) + L(y_2)$ for any differentiable functions y_1 and y_2, then L is a linear operator.
   It also has to satisfy $L(cy) = cL(y)$.
   (f) [T] The isoclines of the DE $y' = f(x, y)$ are the level curves of the function $f(x, y)$.
   By definition of what an isocline is.
   (g) [F] The DE $e^xy' + e^{x+y} = e^{2x}$ is separable.
   Divide both sides by e^x, then notice that $y' = e^x - e^y$ is not separable.
   (h) [T] If y is an equilibrium solution of a first-order DE, then $y'(x) = 0$ for all x.
   An equilibrium solution is constant, hence its derivative is 0.
   (i) [F] Let L be a linear differential operator and y a solution of the DE $L(y) = \sin^2(x)$. Then cy is also a solution of the same DE for any constant c.
   Only true for homogeneous linear DEs. $L(cy) = cL(y) = c\sin^2(x) \neq \sin^2(x)$ in this case.
   (j) [T] The DE $y' - (x^3 - 1)e^y = 0$ has no equilibrium solutions.
   e^y is never 0.

2. (10 pts) Find the equation of the tangent plane to the function $f(x, y) = e^{x^2} + x^y$ at the point $(1, 3)$.

$$\begin{aligned} f_x(x,y) &= 2xe^{x^2} + yx^{y-1} \implies f_x(1,3) = 2e + 3 \\ f_y(x,y) &= x^y \ln x \implies f_x(1,3) = 0 \end{aligned}$$

So $z = (2e + 3)x + c$, and

$$(2e + 3) \cdot 1 + c = f(1, 3) = e + 1 \implies c = -e - 2.$$

So the equation of the plane is

$$z = (2e + 3)x - e - 2.$$

3. (10 pts) In this problem, you will use Euler's method with a stepsize of $\Delta x = 0.5$ to approximate the solution of the IVP

$$\frac{dy}{dx} = 2x - y, \qquad y(2) = 1.$$

Fill out the table below:

The key to Euler's method is the linear approximation $\Delta y \approx \frac{dy}{dx}\Delta x$. You know $\frac{dy}{dx} = 2x - y$.

x	y	$2x-y$	Δy
2	1	3	1.5
2.5	2.5	2.5	1.25
3	3.75	2.25	1.125

4. (15 pts) In this problem, you will find solutions of the DE

$$\frac{dy}{dx} = \frac{(y^3 - y)2x\ln x}{3y^2 - 1} \qquad (x > 0).$$

(a) Find the equilibrium solutions if there are any.
Set

$$0 = \frac{dy}{dx} = \frac{(y^3 - y)2x\ln x}{3y^2 - 1}$$

and solve for y. The only way for the quotient on the right to be 0 is for $y^3 - y = 0$. (Remember we are solving for y not x.) So

$$y(y+1)(y-1) = 0 \implies y = 0, \pm 1.$$

(b) Use separation of variables to find all the nonequilibrium solutions. You may not be able to solve for y explicitly.

$$\int \frac{3y^2 - 1}{y^3 - y}\,dy = \int 2x\ln x\,dx$$

Evaluate the LHS by substituting $u = y^3 - y$, so $du = 3y^2 - 1dy$:

$$\int \frac{3y^2 - 1}{y^3 - y}\,dy = \int \frac{du}{u} = \ln|u| + C = \ln|y^3 - y| + C,$$

and the RHS by integration by parts: $u = \ln x$ and $dv = 2xdx$, so $du = \frac{1}{x}dx$ and $v = x^2$:

$$\int 2x\ln x\,dx = x^2\ln x - \int x^2\frac{1}{x}\,dx = x^2\ln x - \frac{x^2}{2} + C.$$

Set these equal and combine arbitrary constants into C_1:

$$\begin{aligned}
\ln|y^3 - y| &= x^2\ln x - \frac{x^2}{2} + C_1 \\
|y^3 - y| &= e^{x^2\ln x - \frac{x^2}{2} + C_1} \\
y^3 - y &= \pm e^{C_1} e^{x^2\ln x - \frac{x^2}{2}} \\
y^3 - y &= C_2 e^{x^2\ln x - \frac{x^2}{2}} \qquad (C_2 \neq 0)
\end{aligned}$$

That's about the best we can do to solve for y as an implicit function of x.

(c) Check your solution by differentiating it. (Hint: you may have to use implicit differentiation.)
Differentiate both sides of the solution with respect to x. Keep in mind that $y = y(x)$, so we need to use implicit differentiation:

$$
\begin{aligned}
3y^2\frac{dy}{dx}-\frac{dy}{dx} &= C_2 e^{x^2\ln x-\frac{x^2}{2}}\left(2x\ln x - x^2\frac{1}{x}-x\right)\\
(3y^2-1)\frac{dy}{dx} &= \underbrace{C_2 e^{x^2\ln x-\frac{x^2}{2}}}_{y^3-y}(2x\ln x)\\
\frac{dy}{dx} &= \frac{y^3-y}{3y^2-1}\,2x\ln x \qquad \checkmark
\end{aligned}
$$

(d) Find the particular solution that fits the initial condition $y(1)=2$.

$$
\begin{aligned}
2^3-2 &= C_2 e^{1^2\ln 1-\frac{1^2}{2}}\\
6 &= C_2 e^{-\frac{1}{2}} \implies C_2 = 6e^{\frac{1}{2}}
\end{aligned}
$$

So the particular solution is

$$
y^3-y = 6e^{x^2\ln x-\frac{x^2}{2}+\frac{1}{2}}.
$$