Math 3C Practice Exercises

Lecture 2, W'03, Mar 11, 2003

NB: the final exam is 7:30–10:30 PM on Mon, Mar 17 in Girvetz 1004.

Unless told otherwise, your answers must be carefully justified. It is not enough to have a correct answer, you must explain how you got it. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. Simplify your answers as much as you can. It's ok to leave $\sqrt{3}$, π, e^3 and such, but not 7/14, $\sqrt{9}$, $\log 1$, or $e^{\ln 5}$.

You may not use books, notes, and calculators on the exam, but you may have two 3" by 5" (double-sided) handwritten cheat sheets. Also, bring a pencil, eraser, scratch paper, and your student ID.

Warning: "Learning" the answers to practice problems is nearly–probably completely–useless. If you know how to do all the problems below in the sense that you can do all problems of the genres they represent, no matter what notation is used in the statements and no matter how they are formatted, you are in quite good shape. The problems on the final will look similar to these to someone with a solid understanding of the material; but may look completely different to someone whose knowledge is shallow. In any case, they will be doable with the techniques and ideas needed to solve these. Some of these are more demanding than a version on the final might be. Also, the final will be much shorter (about a third of this).

1. Decide if the following are true or false.
   (a) ☐ Let $f(x, y)$ be a differentiable function. Then $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = 0$.
   (b) ☐ Let f be a continuous function. Then $\dfrac{d}{dx}\displaystyle\int_x^a f(t)\,dt = -f(x)$ for any a.
   (c) ☐ If the isoclines of the DE $\dfrac{dy}{dx} = f(x, y)$ are horizontal lines, then $\dfrac{\partial}{\partial x} f(x, y) = 0$.
   (d) ☐ Let L be a linear operator. Then $L(y_1 y_2) = L(y_1)L(y_2)$.
   (e) ☐ If $y(t)$ is a solution of the DE $y' = f(y)$, where f is some function, then $y(t - c)$ is also a solution for any number c.
   (f) ☐ Let $\dfrac{dy}{dx} = f(y)$. If $f(c) = 0$ and $f'(c) > 0$, then $y = c$ is a stable equilibrium solution of the DE.
   (g) ☐ $\dfrac{d}{dx}\displaystyle\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1)c_{n+1}\, x^n$.
   (h) ☐ $\displaystyle\int \sum_{n=0}^{\infty} c_n x^n\, dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}\, x^{n+1}$.
   (i) ☐ Suppose $\displaystyle\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. Then the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is convergent for any x.
   (j) ☐ Any first-order linear DE is separable.

2. Consider the surface $V = f(u, w) = w + w^2 \cos(uw)$ in (u, w, V) space.
   (a) Draw the w-section $w = 1$ (do this in 2 dimensions, not 3). Label your axes and label a couple of points on the curve with their coordinates.
   (b) Find the equation of the tangent plane at the point $(u, w) = (\pi, 1)$. Your answer should be in the form $V =$ a linear function of u and w.

3. Consider the differential equation $y' = 2ty + 1$. In the same picture:
   (a) Sketch some isoclines.
   (b) Sketch the direction field.

4. Label each of the following differential equations as S (separable), LH (linear and homogeneous), LI (linear and inhomogeneous), A (autonomous), or N (none of the preceding). Use all labels that apply.

(a) $y' = e^{\sin(t)} y - t$
(b) $\dfrac{dw}{ds} + \cos(w^2) e^s = 0.$
(c) $y' = (y - t)^2.$
(d) $z' - \left(e^t - e^{-t}\right) z = 0.$
(e) $y' = e^y + t.$
(f) $y' = \dfrac{1 + t^3}{4 + \cos(y^2)}.$
(g) $y' = y^3 + e^y + 17.$

5. Determine all the equilibrium solutions of the differential equation

$$y' = (y+1) y^2 (y-3)^3 (y-4)^2.$$

Sketch them in a slope field picture for the equation and mark them on the phase line (the line $t = 0$) with an open circle if they are unstable, a filled circle if they are stable and a split circle if they are semistable. Your slope field picture should show why these markings are correct.

6. Consider the differential equation $y' - t\sin(t)(y^3 - 2y^2 + y) = 0$
   (a) Find all equilibrium solutions.
   (b) Find the general solution. You may not be abel to solve for y explicitely.
   (c) Verify that your general solution does solve the equation.
   (d) Find the solution satisfying $y(\pi) = 1/2$.

7. Use $h = 1/4$ to perform four steps of the Euler method to approximate the solution of the initial value problem

$$y' = y, \qquad y(0) = 1.$$

at $t = 1$. The exact value of $y(t)$ at $t = 1$ is $e = 2.7182818285\ldots$, what is the total error of your approximation?

8. Consider the differential equation $y' = \cos(t) y + \sin(t)$
   (a) Verify that $y(t) = e^{\sin(t)} \int_{23}^{t} e^{-\sin(x)} x\, dx$ is solution to the differential equation with $y(23) = 0$. (NOTE: You are not asked to solve the equation and you should not try to explicitly evaluate the integral. You are asked instead to check that the given function satisfies the equation and the given initial condition).
   (b) Find the general solution of the equation. (Hint: You already have a "particular solution.")
   (c) Find the solution which satisfies $y(23) = 115$.

9. Find, by inspection, a particular solution of the equation

$$y' + 8\frac{1}{1+t} y = \frac{1}{1+t}.$$

Then find the general solution of the equation.

10. Let $L(y) = y' + p(t)y$ where $p(t)$ is some function.
   (a) Use the definition of linearity to prove that L is a linear operator.
   (b) Assume that $L(y_1) = t$, $L(y_2) = \cos(t)$, and $L(y_3) = 0$ where y_1, y_2 and y_3 are some functions and $y_3 \neq 0$. What is the general solution of $L(y) = 2t - 3\cos(t)$?

11. Find the solution of the IVP first using variation of parameters, then by the integrating factor method.

$$\frac{dy}{dt} = \sin(t) y + \sin(t)\cos(t), \quad y(1) = 1.$$

Your final answer should be in closed form and contain no unevaluated integrals.

12. Find the solution of the IVP first by the integrating factor method, then using variation of parameters.

$$\frac{dy}{dt} + 4\frac{y}{t} = \frac{e^t}{t^3}, \quad y(1) = 1.$$

Your final answer should be in closed form and contain no unevaluated integrals.

13. The solution of $w' + q(t)w = h(t)$, $w(a) = w_0$ is

$$w(t) = e^{-\int_a^t q(r)\,dr} w_0 + e^{-\int_a^t q(r)\,dr} \int_a^t e^{\int_{t_0}^t q(r)\,dr} h(s)\,ds.$$

(a) Derive this formula.
(b) Show directly that the formula provides a function which satisfies the equation and the initial condition.

14. Suppose the function y satisfies

$$\int_0^{y(t)} \frac{1}{g(s)}\,ds = \int_0^t f(s)\,ds$$

for all t in some interval. Show, using only the Fundamental Theorem of Calculus and the Chain Rule, that then $y'(t) = g(y(t))f(t)$; that is y solves $y' = g(y)f(t)$. Justify any computations by giving the reasons why you think they are correct.

15. Let $x(t)$ be the mass of a radioactive substance at time t.
(a) The rate of decay of a radioactive substance is proportional to the amount present at any time. Use this information to find the differential equation that $x(t)$ satisfies.
(b) Let's say that initially there is a 25 g of radioactive material and it is decaying continuously at a rate of 2%/year. Solve this IVP for x.
(c) What is the half-life of this substance? (You may leave logs in the answer.)

16. Al Capone is blending moonshine in the basement of his secret bar. His apparatus consists of two connected barrels. Newly-distilled whiskey containing 80% alcohol is dripping into barrel 1 at a rate of 10 gal/h. This mixture is constantly stirred and flows out of barrel 1 and into barrel 2 at the same rate. The mixture in barrel 2 is also thoroughly stirred and flows out at the same rate to be bottled. Barrel 1 is 50 gals, barrel 2 is 100 gals. They are initially both filled with old whiskey containing 40% alcohol.
(a) Set up an IVP for the amount of pure alcohol in barrel 1 as a function of time. Solve this IVP.
(b) Do the same for barrel 2.
(c) How long before the whiskey flowing out of barrel 2 contains 50% alcohol?

17. Lt. Columbo is investigating a bank robbery, which took place at 4PM. He visits the suspect's home at 8PM. The suspect's friends assure him that they (incl. the suspect) were watching the Smurfs on video all day long, and they didn't even leave the house since breakfast. On his way out, Columbo notices that the suspect's car feels warm. He measures the temperature of the coolant and it's 25°C. An hour later, he sneaks back to take another measurement. It is now 20°C. The thermostat in the garage is set to 15°C. He knows that the normal coolant temperature in a running engine is 95°C.
(a) Assuming that Newton's Law of Cooling applies, find a differential equation to describe the temperature of the coolant in the car.
(b) Solve the equation, and use the given data to figure out the constants in your solution.
(c) Should Columbo believe the suspect and his friends?

18. A barrel of monkeys is released on a monkey-less tropical island. Five years later, 280 monkeys are found swinging in the trees. Another 5 years later, 700 monkeys are counted on the island. Assume that the monkey population obeys the logistic equation and the island's banana crop can sustain 2800 monkeys.

   (a) Write down the logistic equation and solve it. Show all details of the solution.
   (b) Using the population data and the maximum population, find the particular solution of the equation. (Hint: You can take advantage of the fact that the equation is autonomous to make the initial conditions simple.)
   (c) How many monkeys were in the barrel?

19. (a) Find the Taylor series for $f(x) = 1/x$ at $x = 1$.
   (b) What is the radius of convergence?

20. (a) Find the Taylor series for $\sinh x = (e^x - e^{-x})/2$ at $x = 0$. (Hint: you may use what you know about the Taylor series for e^x.)
   (b) For what x does the Taylor series for $\sinh x$ converge?
   (c) Recall $\cosh x = (e^x + e^{-x})/2$. Use $\frac{d}{dx} \cosh x = \sinh x$ and the result from part (a) to find the Taylor series for $\cosh x$ at $x = 0$. How do you know what the constant term is?
   (d) For what x does the Taylor series for $\cosh x$ converge?

21. Show the following:
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. (Hint: the derivative of $\ln(\ln(x))$ is $1/(x \ln x)$. Use a similar argument to exercise II.6.4 in the Reader.)
   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges. (Hint: the derivative of $1/\ln(x)$ is $-1/(x(\ln x)^2)$. Use a similar argument to exercise II.6.7 in the Reader.)

22. Find the interval of convergence for the following power series.
   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n\,2^n}$.
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$.
   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$.
   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^2} (1-x)^n$.
   (e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n\, n!\, (x+5)^{2n-1}$.

23. **Integration and differentiation review**. You may want to attempt the following exercises from Stewart: 3.6.11-16, Ch. 3 review exercises 27-34, Ch. 7 review exercises 11-13, 16-23, 25-28. You can, of course, do others too for additional practice.