Additional notes for Math 3C by Michael Crandall

1. Functions and Calculus: A Rewind

To me, calculus is mainly about functions. I'd like you to be comfortable with functions like me; it will help you in this course, so we'll chat about them a while. Please don't worry about this material if it confuses you at first (I hope it won't). However, do get us to help you if it does.

So far you have mainly considered "real-valued functions of one real variable". This means a gadget f into which one can plug certain numbers (any number in the "domain" of f) and get out a number. If we plug in x we get out $f(x)$, which is called "the value of f at x" or "f evaluated at x".

We'll talk about real-valued functions of one real variable for a while. We indicate these kinds of functions lots of ways. For example, we might write "x^2" to indicate the function which squares numbers or we might write $y = x^2$ or t^2 or $f(z) = z^2$ and always be talking about the same old function which squares numbers. We don't really need any notation to talk about this simple function: Put in a number, get out its square. Lets call this the "square" function. The square function evaluated at 3 is 9, the value of the square function at -1 is 1, etc.

You have studied some especially important functions. These include the constant functions, the identity function (this means the function given by $f(x) = x$), more general linear functions, polynomials, the one over function (multiplicative inverse), power functions, the trig functions, the exponential function (this means the function given by $f(x) = e^x$), and the logarithm function (and we always mean the "natural logarithm" ln, unless we say otherwise).

Math 3A, wherever you took the analogue of it, is mostly about a particular way to get a new function f', called the derivative of f, from a given function f. This derivative is a really important, amazing tool. For example, the derivative of the exponential function is the exponential function, the derivative of the sine function is the cosine function, the derivative of the logarithm function is the one over function, etc.

You learned how to compute derivatives of functions built up from functions you already knew how to differentiate. For example, if f, g are two real valued functions of a real variable, then the product fg of f and g denotes the function whose value at x is $f(x)g(x)$. Its derivative is given by $(fg)' = f'g + g'f$; that's the product rule.

The killer rule of differentiation theory at this level is the "chain rule". It is about differentiating the function $h(x) = f(g(x))$, called the composition of f and g, aka f composed with g. Sometimes this is written $h = f \circ g$. To evaluate h at x, we first evaluate g at x and then evaluate f at $g(x)$. If f is the cosine function and g is the square function, then $h(x) = \cos(x^2)$. The chain rule says: if h is the composition of f and g, then h' is given by $h'(x) = f'(g(x))g'(x)$. The derivative of f composed with g is the composition of f' and g multiplied by g'. If this doesn't confuse you, you are in good shape with functions. If it does confuse you, we need to work on this a bit. Here is a way to tell: given

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	-11	π	-7
π	15	-23	31	13
-7	17	-11	12	57
6	e	-37	12.8	101

what is the derivative of $h = f \circ g$ evaluated at 1? Can you compute the derivative of h at π from this data, and if not, why not?

If f is a function, then so is f' and then so is $(f')' = f''$ and $f^{(j)}$, denoting f differentiated j times. Which of the following functions: square, cosine, sine, log, exponential, identity, satisfy the following equations?

$$(\text{i})\ f' = f,\ (\text{ii})\ f'' + f = 0,\ (\text{iii})\ f' = 1,\ (\text{iv})\ xf'(x) = 1,$$
$$(\text{v})\ f'' = f^2,\ (\text{vi})\ f'(t) = 2t,\ (\text{vii})\ f^{(13)} = f.$$

For example, does the square function satisfy (i)? No, because if $g(x) = x^2$ is the square function, then $g'(x) = 2x \neq g(x) = x^2$ unless $x = 0$ or $x = 2$. When we write an equation like $f' = f$ saying two functions are the same, it means that they have the same values at every point (in their domains).

The last bit of review we take up here is the main tool studied in Math 3B, the integral. If f is a reasonable (continuous) function whose domain contains an interval $[a, b]$, then

$$F(x) = \int_a^x f(s)\,ds \tag{1}$$

defines a function F by giving its value at each $x \in [a, b]$ as the definite integral from a to x of f. F is well-defined in the mathematical sense by this formula, even if it is impossible to "evaluate" the integral in terms of the functions whose names you know. For example, you cannot write a better formula for $\int_0^x e^{\sin(t)}/(1 + t^2)\,dt$ than the integral notation gives.

The Fundamental Theorem of Calculus (FTC) is built from two main parts. Part 1 says that if F is given by (1), then $F' = f$. That is, the integral solves for us the "differential equation" $F' = f$ when f is given. Part 2 says that if g is any function for which $g' = 0$ (that is, g' is the zero function), then g is a constant function.

We put these two parts together to deduce the FTC:

Theorem 1. *If f is continuous on $[a, b]$, then $F' = f$ if and only if*

$$\int_a^x f(s)\,ds = F(x) - F(a) \quad \textit{for} \quad a \leq x \leq b. \tag{2}$$

Proof. To prove this, we note that if (2) holds, then differentiating the left-hand side yields f by Part 1 above, so $F' = f$. That is, (2) implies $F' = f$. On the other hand, if $F' = f$, then the derivative of $\int_a^x f(s)\,ds - F(x)$ is zero, so it is constant by Part 2. Evaluating at $x = a$, the constant is $-F(a)$, so (2) holds. □

You are used to using the FTC to evaluate integrals, for example $\int_0^7 x^2\,dx = 7^3/3$ because the derivative of $x^3/3$ is x^2. In this course, we are also interested in the other direction, that the integral solves the differential equation, even if we cannot explicitly integrate the thing.

HINT: Stating the FTC could be on a quiz. Soon. Likewise, the Chain Rule, where we mean the form given above. The single equation

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

is an acceptable answer to the chain rule question. However, things like $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ are not. I want you to learn the forms discussed above. You are welcome to use whatever you like in computations however.

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Be aware that this material is provisional and may change a little, check back.

2. Separation of Variables and the Linear Equation

Very important parts of what goes on in Chapters 1 and 2 of the text are first "solving" separable first order differential equations (Chapter 1) and then solving linear first order equations (Chapter 2). There is a lot of yada yada around these two main points, and we want you to realize that you already know all you need to know to solve these problems. This is the spirit of this part of these notes. Some of you will understand what follows at once and then a lot of the rest of the material will be quite easy for you, and some of you will take a while. One of our goals is that everyone should understand this presentation by the end of the course.

The general first order ordinary differential equation has the form

$$y' = f(t, y) \tag{3}$$

where y is the "unknown" function, aka "the dependent variable" aka "the state variable". Here f denotes a given function of two variables. The very meaning of (3) is given by saying when a function y (of one real variable) is a *solution* of it. A function y is a *solution* of (3) on some interval exactly when

$$y'(t) = f(t, y(t)), \text{ equivalently, } \frac{dy}{dt}(t) = f(t, y(t)),$$

for all numbers t in the interval.

In addition, we consider the *initial value problem*:

$$y' = f(t, y) \quad \text{and} \quad y(t_0) = y_0. \tag{4}$$

This is like (3) except that two numbers t_0 (aka the initial time) and y_0 (aka the initial value or the initial state) are given and we seek solutions y of $y' = f(t, y)$ which also satisfy the *initial condition* $y(t_0) = y_0$.

The text and lecture are full of examples, we will just deal with the general case here.

The Case Where $f(t, y)$ Actually Depends Only on t. In this case we treat the very important equation

$$y' = f(t). \tag{5}$$

This means the same as $y' = f$, which makes sense because f is a function of one variable here.

CAUTION: If f is a function of two variables, the symbol f' has NO MEANING in this course (wait till 5B for this).

Aha! We already know all about (5). To "solve" it we just have to find the functions whose derivative is f. This is the problem of integration. If we cannot write down any functions whose derivative is f in closed form, we are still ok. According to the FTC, y is a solution of (5) if and only if

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s)\, ds$$

when t_0, t are any numbers in an interval where y is defined. In particular, the solution of the intial value problem $y' = f(t)$, $y(t_0) = y_0$ is given by the formula

$$y(t) = y_0 + \int_{t_0}^{t} f(s)\, ds. \tag{6}$$

The Case Where "$f(t, y)$" Is a Product $f(t)g(y)$. There are not enough letters in the alphabet to avoid the notational problem we meet here. We introduced the notation $f(t, y)$ in (3) and now we are going to consider the equation

$$y' = f(t)g(y) \tag{7}$$

and the initial value problem

$$y' = f(t)g(y) \quad \text{and} \quad y(t_0) = y_0. \tag{8}$$

in which "f" is used in a *different way*. We call this "abuse of notation", be alert to it and it won't bite you. Note that we already abused notation in the previous case.

The first thing we want to notice is that if $g(y_0) = 0$, then $y \equiv y_0$ (meaning y "is identically" y_0; that is, it is the constant function y_0) is a solution of (8).

<u>Solution of $y' = g(y)f(t)$, when $y(t_0) = y_0$ and $g(y_0) = 0$</u>

If $g(y_0) = 0$, then $y \equiv y_0$ is a solution of (8)

In view of the above, we have handled the case when y is a solution of (7) and $g(y) = 0$ somewhere. We expect then that y is a constant function. Assuming that the algebra is ok (that is, $g(y) \neq 0$), we rewrite (7) as

$$\frac{1}{g(y)} y' = f(t).$$

Suppose now that $G' = 1/g$. Then the chain rule says

$$\frac{d}{dt} G(y(t)) = G'(y(t))y'(t) = \frac{1}{g(y(t))} y'(t) = f(t).$$

Thus, if $F'(t) = f(t)$, then

$$G(y(t)) = F(t) + c$$

for some constant c. This is because then

$$\frac{d}{dt}(G(y(t)) - F(t)) = f(t) - f(t) = 0$$

and only constant functions have the zero function as their derivative. We conclude that

<u>Implicit solution of $y' = g(y)f(t)$ when $g(y(t))$ never vanishes</u>

If $y' = f(t)g(y)$, $G' = 1/g$, and $F' = f$, then for some constant c, $G(y(t)) = F(t) + c$.

The initial-value problem adds the condition $y(t_0) = y_0$. This allows us the determine the constant c:

$$G(y(t_0)) = G(y_0) = F(t_0) + c \quad \text{or} \quad c = G(y_0) - F(t_0).$$

so

<u>Implicit Solution of $y' = g(y)f(t), \;\; y(t_0) = y_0$ when $g(y_0) \neq 0$</u>

$$G(y(t)) = F(t) + G(y_0) - F(t_0).$$

There is a main point here, and it is not about magic with "differentials". The point is that any expression of the form $h(y)y'$ is a derivative of a function of y. If $H' = h$, then

$$\frac{d}{dt} H(y(t)) = H'(y(t))y'(t) = h(y(t))y'(t).$$

The Case Where "$f(t, y)$" is $f(t) - p(t)y$. Note the abuse of notation. Here we rewrite

$$y' = f(t) - p(t)y$$

as

$$y' + p(t)y = f(t) \tag{9}$$

To solve this one, we take a function P with $P' = p$. Then mulitply (9) by $e^{P(t)}$ to find

$$e^{P(t)}y'(t) + e^{P(t)}p(t)y(t) = e^{P(t)}f(t).$$

Now notice that $\frac{d}{dt}e^{P(t)} = e^{P(t)}P'(t) = e^{P(t)}p(t)$ so the above says

$$e^{P(t)}\frac{dy}{dt}(t) + y(t)\frac{d}{dt}\left(e^{P(t)}\right) = \frac{d}{dt}\left(e^{P(t)}y(t)\right) = e^{P(t)}f(t).$$

Then integrate

$$\frac{d}{dt}\left(e^{P(t)}y(t)\right) = e^{P(t)}f(t)$$

from t_0 to t to find

$$\int_{t_0}^{t}\frac{d}{ds}\left(e^{P(s)}y(s)\right)\,ds = e^{P(t)}y(t) - e^{P(t_0)}y(t_0) = \int_{t_0}^{t}e^{P(s)}f(s)\,ds.$$

Here the first equality comes from the FTC and the second from the above. Solving for y we find

$$\begin{aligned} y(t) &= e^{-P(t)}e^{P(t_0)}y(t_0) + e^{-P(t)}\int_{t_0}^{t}e^{P(s)}f(s)\,ds \\ &= e^{P(t_0)-P(t)}y(t_0) + \int_{t_0}^{t}e^{P(s)-P(t)}f(s)\,ds. \end{aligned}$$

In all, we have the solution formula for (9): If $P' = p$, then:

<u>The General Solution of $y' + p(t)y = f(t)$: 1st Form</u>

$$y(t) = e^{P(t_0)-P(t)}y(t_0) + \int_{t_0}^{t}e^{P(s)-P(t)}f(s)\,ds$$

We can make this look more complicated by using

$$P(t) - P(\tau) = \int_{\tau}^{t}p(r)\,dr$$

(the FTC again!) with $\tau = t_0$ and with $\tau = s$ to find:

<u>The General Solution of $y' + p(t)y = f(t)$: 2nd Form</u>

$$y(t) = e^{-\int_{t_0}^{t}p(s)\,ds}y(t_0) + \int_{t_0}^{t}e^{-\int_{s}^{t}p(r)\,dr}f(s)\,ds$$

<u>Summary of Tools Used</u> Notice that the basic things one needed to know to understand all of the above are: The Chain Rule, The Fundamental Theorem of Calculus, the Product Rule and Properties of the Exponential Function. That's all. This is pretty good support for my endless badgering to the effect that I really, really want you to own these things.

2.1. **Material Added 2/28/03.**

Remark 2. Why are we talking about "the general solution" above when we are giving formulas which solve the initial value problem? Well, given *any* solution of $y' + p(t)y = f(t)$ it is given by the formulas, so they are "general" in this sense. For example, $y(t) = e^t$ is a solution of $y' - y = 0$ and it satisfies $y(1) = e$ or $y(-3) = e^{-3}$, etc., and hence is given by any of the above formulae with $p(t) = -1$ and $t_0 = 1$ or $t_0 = 3$ and the corresponding value of $y(t_0)$. Try it. Those formulas are "general", they give all solutions.

If you want to write something which appears more like what is in the text as the "general solution", things that have a "c" in them, do this sort of thing: let us say 0 is in the domain of $p(t)$, put $t_0 = 0$ and replace $y(t_0)$ by "c". Putting $P(t) = \int_0^t p(s)\,ds$ and $t_0 = 0$ in the first form above yields the "general solution"

> The General Solution of $y' + p(t)y = f(t)$: Modified 1st Form
>
> $$y(t) = ce^{-\int_0^t p(r)\,dr} + e^{-\int_0^t p(r)\,dr} \int_0^t e^{\int_0^s p(r)\,dr} f(s)\,ds$$

Now if you want to choose c so as to satisfy an initial condition $y(0) = y_0$, just take $c = y_0$. If you want to choose c so as to satisfy some other initial condition, say $y(-1) = 7$, you will have to work. But why bother, why not just use the formula with $t_0 = -1$, $y_0 = 7$?

Are There Any Serious Reasons to Prefer the "theory" above to the text's form

> The General Solution of $y' + p(t)y = f(t)$: Text's Form
>
> $$y(t) = ce^{-\int p(t)\,dt} + e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt} f(t)\,dt$$

Yes, there certainly are. For example, if $p(t) = \sin(t^2)$, $f(t) = 1$, you cannot find any of the antiderivatives indicated in this formula and then you certainly cannot solve for "c" to satisfy an initial condition. You might even think that the initial value problem "has no solutions".

That is, if you understand the linear fode $y' + p(t)y = f(t)$ as instructions to find antiderivatives and do procedures you completely miss the fact that

$$y' + \sin(t^2)y = 1, \quad y(0) = 12$$

has exactly one solution and it is:

$$y(t) = 12e^{-\int_0^t \sin(r^2)\,dr} + e^{-\int_0^t \sin(r^2)\,dr} \int_0^t e^{\int_0^s \sin(r^2)\,dr}\,ds = 12e^{-\int_0^t \sin(r^2)\,dr} + \int_0^t e^{-\int_s^t \sin(r^2)\,dr}\,ds.$$

Those definite integrals make perfect sense in spite of our inability to find antiderivatives here.

Some Other Things From Class Not in the Book. One thing we pointed out is this: if r, d are constants, then to solve the ivp

$$A' = rA + d, \quad A(0) \text{ given}$$

we can rewrite it as

$$(rA + d)' = rA' = r(rA + d)$$

or $y' = ry$ where $y = rA + d$. Then $y(t) = y(0)e^{rt}$ says that

$$rA(t) + d = e^{rt}(rA(0) + d) \implies A(t) = e^{rt}A(0) + d\frac{e^{rt} - 1}{r}.$$

Next, we gave some work saving remarks.

If you see the fode $y' = g(y)$ only as instructions to do procedures to find y's, the meaning of which you don't much care about, you will have forgotten that $y' = g(y)$ is "defined" by saying when a function is a solution of it: a function y is a solution of $y' = g(y)$ if $y'(t) = g(y(t))$ for all t's in the domain of y. Here t is any number, so we can also write $y'(\#) = g(y(\#))$ for any number $\#$ in the domain of y and mean the same thing.

Thus it makes perfect sense to say a function w is a solution of $y' = g(y)$. This means that $w'(t) = g(w(t))$, etc.

If y is a solution of $y' = g(y)$ and τ is an fixed number, then $w(t) = y(r(t+\tau))$ is a solution of $w' = rg(w)$. To see this, notice that $w'(t) = ry'(r(t+\tau)) = rg(y(r(t+\tau))) = rg(w(t))$! The first equality is from the chain rule, the second is from the fact that $y'(\#) = g(y(\#))$ with $\# = r(t+\tau)$. This computation depended on the fact that $y' = g(y)$ is AUTONOMOUS! If $y' = f(t, y)$ instead, we would have

$$w'(t) = ry'(r(t+\tau)) = rf(r(t+\tau), y(r(t+\tau)) = rf(r(t+\tau), w(t))$$

which is more complicated.

Consequence: To solve $w' = rg(w)$, $w(t_0) = w_0$, solve $y' = g(y)$, $y(0) = y_0$ and put $w(t) = y(r(t-t_0))$. By the above, w solves the desired differential equation and also $w(t_0) = y(r(t_0-t_0)) = y(0) = y_0$.

For example, the solution of the problem

$$y' = y\left(1 - \frac{y}{L}\right), \quad y(0) = y_0 \neq 0$$

is

$$y(t) = \frac{L}{1 + \left(\frac{L}{y_0} - 1\right)e^{-t}} \tag{10}$$

Therefore, with no additional work, the solution of

$$y' = ry\left(1 - \frac{y}{L}\right), \quad y(t_0) = y_0 \neq 0 \tag{11}$$

is

$$y(t) = \frac{L}{1 + \left(\frac{L}{y_0} - 1\right)e^{-r(t-t_0)}} \tag{12}$$

and the solution of the ivp for the threshold equation

$$y' = -ry\left(1 - \frac{y}{T}\right), \quad y(t_0) = y_0 \neq 0 \tag{13}$$

is

$$y(t) = \frac{T}{1 + \left(\frac{T}{y_0} - 1\right)e^{r(t-t_0)}}. \tag{14}$$

To get (12) we put $w(t) = y(r(t - t_0))$ where y is from (10) and then called it y again. To get (14) we put $w(t) = y(-r(t - t_0))$ where y is from (10) with T in place of L and then called it y again.

Finally, we observed that if c is an equilibrium (constant) solution of $y' = g(y)$, that is, $g(c) = 0$, then

$$g'(c) > 0 \implies c \text{ is unstable.}$$

This is because g is increasing at c, so for $y > c$ and near c, $g(y) > 0$. Initial values a bit above c are driven away from it by the differential equation (do a "slope field" picture). Similarly, initial values a bit below c are driven down and away from c. Similarly,

$$g'(c) < 0 \implies c \text{ is stable.}$$

If $g'(c) = 0$, we don't know the situation without more analysis. If $g(y) > 0$ for y near c we have semistabilty, etc.