CHAPTER II
SERIES

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ADVICE TO THE STUDENT: You will notice that many exercises refer back to earlier ones. You will therefore need to keep copies of what you turn in, or better still, keep them in a permanent notebook and turn in the copies.

ADVICE TO THE INSTRUCTOR: This chapter assumes that the ODE material from Math 3C has been covered. I've tried to make this material interesting by including applications to ODEs wherever possible.

1. LINEAR APPROXIMATIONS

Although you might not notice it, all of Math 3A is about a single idea: complicated functions can often be approximated, on a small scale anyway, by straight lines. What good is such an approximation? Many textbooks will have a (rather unconvincing) application, something like 'Approximate the square root of 1.037'. In fact, almost everything that happens in calculus is an application of this idea.

For example, one learns that the graph of a function $y = f(x)$ is increasing at a point $x = a$ if the derivative $f'(a)$ is positive. Why is this true? It's because of the linear approximation idea: the graph is increasing exactly if the straight line approximating the graph is increasing. For a line, it's easy to see that it is increasing if the slope is positive. That slope is $f'(a)$.

The point of this chapter is to generalize this idea to higher order 'better-than-linear' approximations. To do this we will need to start by reviewing the case of linear approximations. There are a lot of different ways to specify a line in the plane by an equation. For us the most useful will be the 'point-slope' form: the line through the point (a, b) with slope m has the equation $y - b = m(x - a)$, or $y = b + m(x - a)$. If $y = f(x)$ is some function, the line through the point $(a, f(a))$ with slope $f'(a)$ is just $y = f(a) + f'(a)(x - a)$. So, if x is close to a,

$$f(x) \approx f(a) + f'(a)(x - a). \tag{1}$$

APPLICATION: Suppose you want to study the motion of a pendulum. The important thing to know is the angle $\theta(t)$ that the pendulum makes with the vertical, as a function of time t. Newton's second law and some consideration of vectors leads to the conclusion that the second derivative $\theta''(t)$ should be equal to $-\sin(\theta(t))$:

$$\theta''(t) = -\sin(\theta(t)).$$

(You will see this formula derived in Math 5A.) It is extremely difficult to find a function $\theta(t)$ that does this. Instead we can approximate the function $-\sin(x)$ near $x = 0$ by a simpler function, a line. Using the formula (1) above with $f(x) = -\sin(x)$ and $a = 0$ we get $f(0) = 0$ and $f'(0) = -\cos(0) = -1$. So $-\sin(x) \approx -x$ if x is not too big. Then an approximation to our original problem is to seek a function $\theta(t)$ which satisfies

$$\theta''(t) = -\theta(t).$$

If the pendulum does not oscillate too much, then $\theta(t)$ will be small, and it is safe to use the approximation to the sin function. The point here is that it is much easier to find a function that behaves this way; $\theta(t) = \cos(t)$ works, so does $\theta(t) = \sin(t)$ or any combination of the two. What we have done here is *linearize* the differential equation.

1.1. Find the equation of the line tangent to $y = e^x$ at $x = 0$. Be sure your answer really is the equation of a line; if not, it's wrong. Plot the line together with e^x on your calculator.

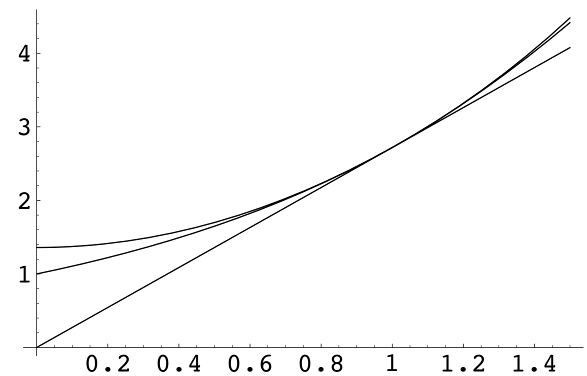
1.2. Find the equation of the line tangent to $y = \sqrt{x}$ at $x = 1$. Plug $x = 1.037$ into the equation for the line; this gives a good approximation to $\sqrt{1.037}$. It's still not a very convincing application. We'll do better in exercise 4.6 below.

1.3. Find the equation of the line tangent to $y = -\sin(x)$ at $x = \pi$. Use this to write the equation describing the motion of a pendulum when it is nearly upside down. Show that $\theta(t) = \cos(t)$ is no longer a solution, but $\theta(t) = \pi + e^t$ is a solution.

2. QUADRATIC APPROXIMATIONS

Suppose we want an approximation to $y = f(x)$ at $x = a$ that is a little better. We might look for a quadratic polynomial. If we want it to be at least as good as the linear approximation, it should touch the graph (i.e., pass through the point $(a, f(a))$), and also be tangent to the graph, so the first derivative should be $f'(a)$. But we also want the second derivatives to match up; the second derivative

Figure 1



of the quadratic should be $f''(a)$. Here's the formula, which a little thought will show does what it is supposed to:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2. \tag{2}$$

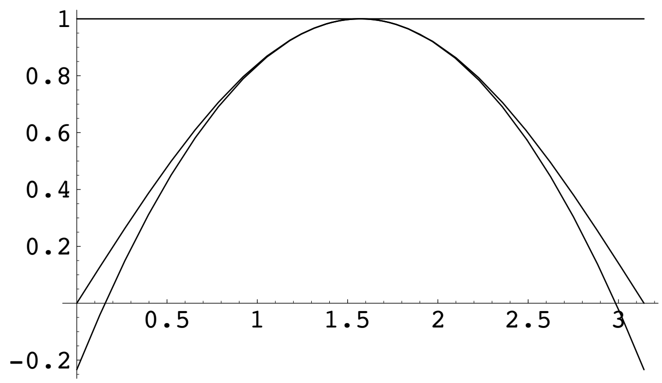
Notice that $f(a)$, $f'(a)$, and $f''(a)$ are all numbers, so the right side really is a quadratic polynomial, while the left side can be a more complicated function. Compare this formula to the one for the linear approximation, (1). All we have done is add an extra term, the $\frac{1}{2}f''(a)(x-a)^2$. This extra term has the property that it is zero at $x = a$, and the derivative of this term is just $f''(a)(x-a)$, also is zero at $x = a$. So adding it on doesn't screw up any of the linear approximation. On the other hand, taking two derivatives of the linear approximation gives zero (check this), while two derivatives of our extra term is exactly what we want: $f''(a)$. The $\frac{1}{2}$ is there to compensate for the fact that taking derivatives will introduce a factor of 2 in that term.

EXAMPLES: Take $f(x) = e^x$, and $a = 1$. Since $f(1)$, $f'(1)$, and $f''(1)$ are all equal to e, we get that

$$e^x \approx e + e(x-1) + \frac{e}{2}(x-1)^2,$$

for x close to 1. Notice that the right side is a quadratic polynomial with the property that its value at $x = 1$, and its first and second derivatives at $x = 1$, are all equal to e. Figure 1 shows the graph of $y = e^x$, together with the linear approximation $y = e+e(x-1)$ (on the bottom) and the quadratic approximation $y = e+e(x-1)+e/2(x-1)^2$ (on the top). Notice the quadratic is much closer to e^x near $x = 1$, but curves away eventually.

Figure 2



Next take $f(x) = \sin(x)$, and $a = \pi/2$. We have $\sin(\pi/2) = 1$, $\cos(\pi/2) = 0$, and $-\sin(\pi/2) = -1$. So

$$\sin(x) \approx 1 - \frac{1}{2}(x - \pi/2)^2$$

for x near $\pi/2$. Figure 2 shows the graph of $y = \sin(x)$, together with the linear approximation $y = 1 = 1 + 0(x - \pi/2)$, and the quadratic approximation $1 - 1/2(x - \pi/2)^2$

WARNING: The most common mistake is to forget to evaluate at $x = a$ at the appropriate time. This gives a result that is not a quadratic polynomial. For example, if you don't plug in $a = \pi/2$, you would write something like $\sin(x) + \cos(x)(x - \pi/2) - \frac{1}{2}\sin(x)(x - \pi/2)^2$. You get an ugly mess, not a nice simple polynomial. Don't do this.

2.1. Find a quadratic approximation to e^x at $x = 0$. Plot the quadratic together with e^x on your calculator. Compare your answer to the example above; the point here is that which polynomial is the best approximation depends on the base point a. You should also compare your answer to the answer to exercise 1.1

2.2. Find a quadratic approximation to $\ln(x)$ at $x = 1$. Plot the quadratic together with $\ln(x)$ on your calculator.

2.3. Find a quadratic approximation to $1/(1 - x)$ at $x = 0$. Plot the quadratic together with $1/(1 - x)$ on your calculator.

2.4. Find a quadratic approximation to $\cos(x)$ at $x = 0$.

2.5. Find a quadratic approximation to $\sin(x)$ at $x = 0$. Sometimes the linear approximation *is* the best quadratic approximation.

2.6. Find a quadratic approximation to $3x^2 + 7x - 4$ at $x = 0$. Draw some conclusion about your answer. Now find the quadratic approximation at $x = 1$. Notice that your answer is really just $3x^2 + 7x - 4$, written in a funny way.

3. HIGHER APPROXIMATIONS

Having carefully read the explanation of how the quadratic approximation works, you should be able to guess the formula to approximate a function with a degree 3 polynomial. (If not, go back and re-read that part of section 2.) What we need to do is keep the quadratic approximation, and just add on a term that makes the third derivatives match up:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3$$

when x is close to a. The factor of $\frac{1}{6}$ is there because the third derivative of $(x-a)^3$ is exactly $6 = 2 \cdot 3$. The notation for higher derivatives is starting to get confusing, so from now on we will also write $f^{(n)}(a)$ for the n-th derivative of $f(x)$ at $x = a$. Remember that $f^{(0)}(a)$ means no derivatives at all, it's just the original function $f(x)$ evaluated at $x = a$.

We can get a polynomial of any degree n to approximate a function $f(x)$, which we will call the n-TH DEGREE TAYLOR POLYNOMIAL OF f AT a

$$f(x) \approx f^{(0)}(a) + f^{(1)}(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n \tag{3}$$

for x near a. You've probably seen $n!$ before, it is the product of all the integers up to n. For example, $4!$ is just $4 \cdot 3 \cdot 2 \cdot 1 = 24$. Notice $1! = 1$, and we just define $0! = 1$, so every term in the polynomial is of the same form,

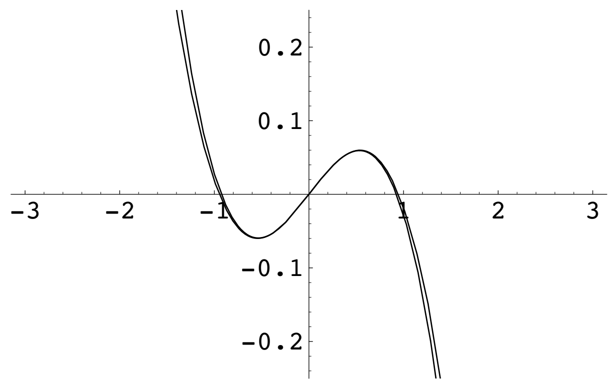
$$\frac{1}{k!}f^{(k)}(a)(x-a)^k,$$

where k goes from 0 up to n.

APPLICATION: Imagine a vertical steel beam, with some load λ (in tons) on it. If it fails, it will start to bend in the middle, and the top half makes some angle x with the vertical. We want to know the angle x as a function of time t (in seconds). From considering force vectors, $x(t)$ satisfies the ODE

$$x'(t) = \lambda \sin(x) - x.$$

Figure 3



This is too hard to deal with, so instead approximate $\lambda \sin(x) - x$ for x near 0, with a degree 3 Taylor polynomial. Suppose, for example, that $\lambda = 7/6$. Then for x near 0 we have

$$\frac{7}{6}\sin(x) - x \approx \frac{x}{6} - \frac{7x^3}{36}.$$

This gives you a simpler model for the beam:

$$x'(t) = \frac{x}{6} - \frac{7x^3}{36}.$$

Figure 3 shows the graph of $y = 7/6\sin(x) - x$, together with the cubic approximation $y = x/6 - 7x^3/36$.

We can analyze this autonomous ODE with the techniques of section 2.5 of the Math 3C text. A little algebra shows that $x/6 - 7x^3/36 = 0$ has three solutions: 0, and $\pm\sqrt{6/7}$. A little calculus shows the graph of $y = x/6 - 7x^3/36$ is decreasing at both $\pm\sqrt{6/7}$, and increasing at 0. So $x = \pm\sqrt{6/7}$ are *stable* equilibria, while $x = 0$ is *unstable*. If the beam is vertical (unbent) at time $t = 0$, the beam will bend, and remain in one of the two bent positions $x = \pm\sqrt{6/7}$.

3.1. Compute the degree 3 Taylor polynomial of each of the functions in the exercises in section 2. In each case, graph the polynomial with the function being approximated.

3.2. Try to compute the degree three Taylor polynomial at $x = 0$ for the function $x^2\cos(3x)$. It is very painful exercise. The next section will show you some very useful shortcuts.

3.3. In the example with the beam at the end of this section, consider what happens if the load $\lambda = 5/6$. Does it bend? In general, how large a load λ is it safe to put on the beam?

3.4. Newton's Law of Cooling says that the rate of change of temperature T with respect to time t is proportional to the difference between the temperature T of the object and the temperature M of the surrounding medium:

$$\frac{dT}{dt} = k_N(M - T), \quad k_N > 0 \text{ the constant of proportionality.}$$

Stephan's Law of Radiation says the rate of change is proportional to the difference of the fourth powers:

$$\frac{dT}{dt} = k_S(M^4 - T^4), \quad k_S > 0 \text{ the constant of proportionality.}$$

Compute the fourth degree Taylor polynomial of $M^4 - T^4$, as a function of T, around the point $T = M$. (M is fixed.) Your answer should show that for T close to M, Newton's Law is a good approximation to Stephan's Law, and tell you the relation between the constants k_N and k_S.

4. POWER SERIES

So far we've discussed how to get various approximations to a function $f(x)$ at a base point a. For example, in exercises 1.1, 2.1, and 3.1 with $f(x) = e^x$ you computed that

$$\begin{aligned} e^x &\approx 1 + x \\ &\approx 1 + x + \frac{x^2}{2} \\ &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned}$$

for x near 0. What we have not done is said how near 'near' has to be, or how close the polynomial is to the function when we use the symbol $\approx$. The two questions are clearly related. We will save these questions for later, however.

In this section we will define TAYLOR SERIES, and, more generally POWER SERIES. Think of the Taylor series as a polynomial of infinite degree, that goes on forever. The coefficients are determined by the formula (3). For example, every derivative of e^x is e^x, and $e^0 = 1$. So the Taylor series for e^x at $x = 0$ is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

This is just a way to keep track of the fact that Taylor approximations of all possible degrees exist for this function, even though in any given computation we would use one of finite degree. Analogously, the decimal digits of $\pi = 3.141592653589793\ldots$ go on forever, even though we can only use finitely many of them in a given calculation.

Another way to write Taylor series more compactly is with the Σ notation, the Greek letter for S (meaning 'sum'). The Taylor series for e^x at $x = 0$ is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{4}$$

The Σ notation just means add up all terms of the form $x^n/n!$, for all n going from 0 to infinity.

The functions $\sin(x)$ and $\cos(x)$ have a nice pattern in their Taylor polynomials, because the derivatives repeat after the fourth one: $\sin(x)$, $\cos(x)$, $-\sin(x)$, $-\cos(x)$, $\sin(x)$, $\cos(x)$, etc. When we plug in 0 we get a sequence $0, 1, 0, -1, 0, 1, 0, -1, \ldots$. So by formula (3) the Taylor series for $\sin(x)$ at $x = 0$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{5}$$

and the Taylor series for $\cos(x)$ at $x = 0$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{6}$$

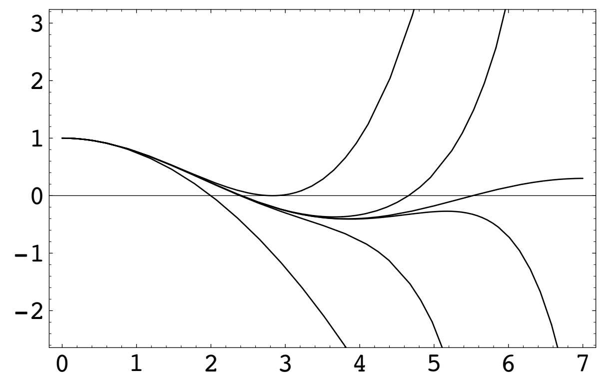
Another basic function that is very useful is $(1-x)^{-1}$. Convince yourself that the n-th derivative of this function is $n!(1-x)^{-n-1}$. So the Taylor series for $(1-x)^{-1}$ is just

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n. \tag{7}$$

The Taylor series (4), (5), (6), and (7) come up so often that you will certainly need to know them.

We use the name TAYLOR SERIES when we start with a function, like e^x, and construct a sequence of polynomials based on it. More generally, we can start by writing any polynomial of infinite degree. This is called a POWER SERIES. A Taylor series is just a special case of a power series.

Figure 4



For example, consider

$$(8)\quad \frac{1}{(0!)^2} - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \cdots \\ \cdots + (-1)^n \frac{(x/2)^{2n}}{(n!)^2} + \cdots = \\ \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n}}{(n!)^2}.$$

We will eventually use these to define new functions. The power series defined by (8) is the Bessel function, $J_0(x)$, with applications to the physics of heat conduction. Figure 4 shows the degree 2,4,6,8, and 10 Taylor approximation to $J_0(x)$, together with the function itself. Since any polynomial eventually tends to infinity, the approximations all eventually curve away from $J_0(x)$. But the higher degree ones 'last longer'. The function $J_0(x)$ itself tends to 0 as x tends to infinity.

WHY ARE WE DOING THIS? It seems a little abstract. Here's the reason: In the previous sections, when computing Taylor polynomials, we had to take a lot of derivatives. This can get very tedious, and it is hard to do accurately. (Recall exercise 3.2). The power series provide a convenient notation for learning the shortcuts to these computations.

EXAMPLES: We can add power series together, multiply, or make substitutions. This will give the Taylor series of new functions from old ones. For example, to get the Taylor series of $\sin(2x)$, we just take

(5) and substitute '$2x$' for 'x'. Thus the series for $\sin(2x)$ is

$$2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots + (-1)^n \frac{2^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}x^{2n+1}}{(2n+1)!}.$$

Similarly if we want the series for e^{x^3}, we take (4) and substitute 'x^3' for 'x'. Thus the series for e^{x^3} is

$$1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \cdots + \frac{x^{3n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}.$$

We can also add or subtract series. The series for $e^x - 1 - x$ is

$$\frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=2}^{\infty} \frac{x^n}{n!}.$$

since $1 + x$ is its own Taylor series (see exercise 2.6). To get the series for $x^2 e^x$, just multiply every term in the series for e^x by x^2, to get

$$x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \cdots + \frac{x^{n+2}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}.$$

MORAL: It is almost always easier to start with a known series and use these shortcuts, than it is to take lots of derivatives and use (3).

4.1. Use (7) to compute the Taylor series at $x = 0$ for each of these functions: $(1 + x)^{-1}$, $(1 - x^2)^{-1}$, $(1 + x^2)^{-1}$.

4.2. Use the methods of this section to compute the Taylor series at $x = 0$ of $x^2 \cos(3x)$. Compare to what you did in exercise 3.2. Re-read the moral at the end of this section.

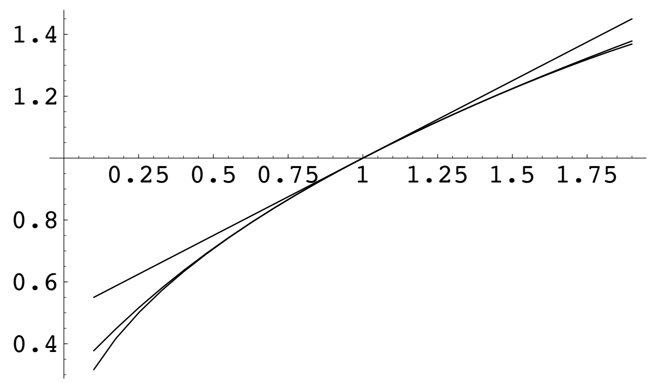
4.3. Compute the Taylor series at $x = 0$ of

$$\frac{1}{2 - x} = \frac{1}{2} \cdot \frac{1}{1 - (x/2)} \quad \text{and of} \quad \frac{1}{2 - x^3}.$$

4.4. Compute the Taylor series at $x = 0$ of $\sinh(x) = (e^x - e^{-x})/2$, $x/(x + 1)$, xe^{-x}, and $x^2 \sin(x^2)$.

4.5. Compute the Taylor series at $x = 0$ of $\sin(x)/x$ and $(e^x - 1)/x$.

Figure 5



4.6. Consider the function $f(x) = (1+x)^t$, where t is some real number. We will use this function to define the BINOMIAL SERIES. Notice $f^{(0)}(0) = f(0) = 1$, and for $k > 1$ the kth derivative at 0 is

$$f^{(k)}(0) = \underbrace{t(t-1)(t-2)\cdots(t-k+1)}_{\text{exactly } k \text{ terms}}.$$

Thus if we define

$$\binom{t}{0} = 1, \quad \binom{t}{k} = \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!},$$

then the Taylor series for $(1+x)^t$ is exactly $\sum_{k=0}^{\infty}\binom{t}{k}x^k$.

Write out explicitly $\binom{4}{2}$, $\binom{4}{5}$, and $\binom{\pi}{4}$. When you multiply it out, $\binom{\pi}{4}$ is about 0.0453135.

It becomes clear that if t is an *integer*, and $k > t$, then $\binom{t}{k} = 0$. In this case $f(x)$ is a polynomial, and its Taylor series is finite. These binomial coefficients $\binom{t}{k}$ may be familiar to you, from Pascal's triangle. If t is *not* an integer, the Taylor series has infinitely many terms.

Write out the first four terms for the binomial series for $(1+x)^{1/2}$, i.e. $t = 1/2$. This is, of course, just the degree 4 Taylor polynomial. Now plug in $x = .037$ to get a good approximation to $\sqrt{1.037}$. Compare this to what your calculator gives, and to your answer to exercise 1.2. Figure 5 shows the function $y = \sqrt{x}$ together with the linear approximation from exercise 1.2 and the degree 4 approximation above.

4.7. Use the binomial series with $t = -1/2$ to find the Taylor series for $1/\sqrt{1-x^2}$.

5. MORE SHORTCUTS

In the previous section we saw how to add, subtract, and make substitutions in power series. In this section we will take derivatives, integrate, and multiply.

Derivatives are easy, as you already know the rules for derivatives of polynomials. Thus, since the derivative of $(1-x)^{-1}$ is $(1-x)^{-2}$, we get the Taylor series for $(1-x)^{-2}$ by taking the derivative of every term in (7):

$$\frac{d}{dx}(1+x+x^2+x^3+\cdots+x^n+\cdots) =$$
$$0+1+2x+3x^2+\cdots+nx^{n-1}+\cdots =$$
$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n.$$

Notice there are two ways to write the series in Σ notation, by keeping track of the power of x, or by keeping track of the coefficient. Since the derivative of e^x is e^x, we should have that the Taylor series (4) is its own derivative, and it is:

$$\frac{d}{dx}(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots+\frac{x^n}{n!}+\cdots) =$$
$$0+1+x+\frac{x^2}{2}+\cdots+\frac{nx^{n-1}}{n!}+\cdots.$$

Every term shifts down by one. Notice that $n/n!$ is just $1/(n-1)!$.

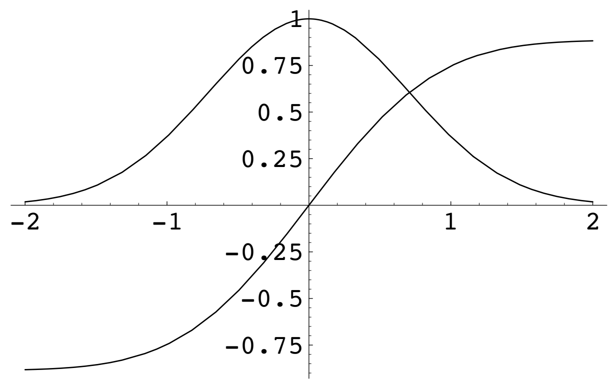
We can also integrate power series 'term by term'. For example, since the antiderivative of $(1-x)^{-1}$ is $-\ln(1-x)$, we get the Taylor series for $-\ln(1-x)$ by computing the antiderivative of each term in (7):

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}\cdots = \sum_{n=1}^{\infty}\frac{x^n}{n}.$$

If we want instead $\ln(1-x)$, of course we have to multiply every term by -1. There is a subtle point here; the antiderivative of a function is determined only up to a constant (the '$+C$' term of Math 3B). In this example, $-\ln(1-x)$ is the unique choice of antiderivative that is zero at $x=0$. That value of the function determines the constant term of the series expansion. So in this case, the constant term is 0.

This idea lets us get a handle on functions that do not have a simple antiderivative. For example, you may have learned in Math 3B that the function e^{-x^2} has no simple antiderivative. The graph of this

Figure 6



function is the 'bell shaped' curve, with applications in statistics. If we take a Taylor series for e^{-x^2}:

$$1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

and integrate term by term we get:

$$x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}.$$

This is the Taylor series for the antiderivative of e^{-x^2} which is 0 at $x = 0$. The function which has this as its Taylor series is called the 'Error function' $\text{Erf}(x)$. By the Fundamental Theorem of Calculus, another way to write this is

$$\text{Erf}(x) = \int_0^x e^{-t^2} dt$$

since $\text{Erf}(x)$ is just the function that computes area under the bell curve up to a variable point x. We have renamed the variable of integration t to keep it separate. Another way to think about this is that we have used power series to solve the differential equation

$$y' = e^{-x^2}, \quad y(0) = 0.$$

Figure 6 shows the graph of $y = e^{-x^2}$, (the 'bell curve') together with the graph of $y = \text{Erf}(x)$ discussed above. Convince yourself by comparing the graphs that the bell curve has the right properties to be the derivative of the Error function.

Here's the last trick of this section: Instead of multiplying a power series by a number, or a power of x, we can multiply two different series together. In general the coefficients of the product are complicated, so we will not use the Σ notation to write the general coefficient; we will just write the first few. For example, the series for $\sin(x)\cos(x)$ comes from multiplying

$$(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots)(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots)$$
$$=x-(\frac{1}{2!}+\frac{1}{3!})x^3+(\frac{1}{5!}+\frac{1}{2!3!}+\frac{1}{4!})x^5+\cdots=$$
$$x-\frac{2}{3}x^3+\frac{2}{15}x^5+\cdots.$$

Notice there are two ways of getting an x^3 term in the series for $\sin(x)\cos(x)$: one from the product of the x term in $\sin(x)$ and the x^2 term in $\cos(x)$, and another from the product of the constant term in $\cos(x)$ and the x^3 term in $\sin(x)$. There are three ways of getting an x^5 term, etc.

5.1. Compute the Taylor series at $x = 0$ for $(1-x)^{-3}$.

5.2. Compute the Taylor series at $x = 0$ for $\ln(1+x)$, $\ln(1-x^3)$, and $\ln((1+x)/(1-x))$. (Hint: These are easy, don't make them into hard problems.)

5.3. In exercise 4.1 you computed the Taylor series for $1/(1+x^2)$. Use this to compute the Taylor series for $\arctan(x)$

5.4. Compute the Taylor series at $x = 0$ for $\arcsin(x)$. (Hint: exercise 4.7.)

5.5. In exercise 4.5 you computed the Taylor series for $(e^x-1)/x$. Use the techniques of this section to find the Taylor series for the 'Exponential integral' function $\text{Ei}(x)$ defined by

$$\text{Ei}(x)=\int_0^x \frac{e^t-1}{t}dt.$$

In other words, $\text{Ei}(x)$ is the antiderivative of $(e^x-1)/x$ which is 0 at $x = 0$. This is another function which comes up in physics and engineering.

5.6. Euler's dilogarithm $L(x)$ is defined by

$$L(x) = \int_0^x \frac{-\ln(1-t)}{t} dt,$$

in other words, it is the antiderivative of $-\ln(1-x)/x$ which is 0 at $x = 0$. It has applications in my area of research, number theory. Compute the Taylor series expansion for $L(x)$ at $x = 0$.

5.7. Compute the first few terms of the Taylor series at $x = 0$ for $\sin(x)e^{2x}$, and for $e^{-x}/(1+x)$.

5.8. The differential equation

$$y' - y = \frac{1}{1+x}, \quad y(0) = 1$$

is first order linear, so the methods of section 2.2 of the Math 3C text should solve it. A difficulty arises in the form of an integral you can't do. Use your answer to the previous problem to find the first few terms of the Taylor series for the solution. (There are other, more systematic ways to find the solution as a series, done in Math 5C.)

5.9. In addition to multiplying series together, one can also do long division. Divide the series for $\cos(x)$ into the series for $\sin(x)$ to get the first few terms of the series for $\tan(x)$. It starts $x + \frac{x^3}{3} + \cdots$, what is the next nonzero term? The series for $\sec(x)$ comes from dividing the series for $\cos(x)$ into $1 = 1 + 0x + 0x^2 + \cdots$. The constant term is 1, what are the next two nonzero terms? You can check your answer by also using formula (3).

5.10. The point of these sections was to develop shortcuts for computing Taylor series, because it is hard to compute a lot of derivatives. We can turn this around, and use a known Taylor series at $x = a$ to tell what the derivatives at $x = a$ are: the n-th coefficient is n-th derivative at a, divided by $n!$. Use this idea and your answer to exercise 4.2 to compute the 10th derivative of $x^2\cos(3x)$ at $x = 0$.

5.11. Try to show that the power series $J_0(x)$ given by formula (8) satisfies Bessel's equation:

$$J_0''(x) + \frac{1}{x}J_0'(x) + J_0(x) = 0$$

by collecting all like powers of x on the left side, and showing everything cancels.

6. CONVERGENCE

In the previous sections we've dealt with power series as formal objects, as way to keep track of all the different Taylor polynomial approximations of all different degrees. We now need to think about what happens for a particular numerical value of the variable x. Do the polynomial approximations get close to the function value? Does it make sense to add up infinitely many numbers? The answer to both question is 'Sometimes.'

We start with the GEOMETRIC SERIES. This is the name for the Taylor series

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n.$$

for the function $1/(1-x)$. This means that for any n, the nth Taylor polynomial at $x = 0$ for $1/(1-x)$ is $1 + x + x^2 + \cdots + x^n$. For a given numerical value of x, we can ask, 'Is this close to $1/(1-x)$? This is true exactly when $(1-x)(1 + x + x^2 + \cdots + x^n)$ is close to 1. A little algebra shows that in the product, a lot of terms cancel; in fact, all but the first and the last:

$$(1-x)(1 + x + x^2 + \cdots + x^n) = 1 - x^{n+1}.$$

This makes it easy to tell what is going on; if $|x| < 1$, then $x^{n+1} \to 0$ as $n \to \infty$. So in this case $(1-x)(1 + x + x^2 + \cdots + x^n)$ is close to 1 for n big, or $1 + x + x^2 + \cdots + x^n$ is close to $1/(1-x)$.

But if $x > 1$, $x^{n+1} \to \infty$ as $n \to \infty$, so $1 - x^{n+1}$ is not close to 1. In this case the Taylor polynomials no longer approximate the function, no matter how big the degree is. (Similar considerations apply when $x < -1$.)

We wish to do a similar kind of analysis for other power series, for which we need to introduce some terminology. Suppose $\{c_n\}$ is any infinite list of numbers (a SEQUENCE). When we write $\sum_{n=0}^{\infty} c_n$ (an INFINITE SERIES or just a SERIES) we mean a *new* list of numbers $\{s_n\}$ formed by taking partial sums

$$\begin{aligned} s_0 &= c_0 \\ s_1 &= c_0 + c_1 \\ s_2 &= c_0 + c_1 + c_2 \\ &\vdots \end{aligned}$$

For example, take the geometric series above, with $x = 1/2$. So c_n will be $(1/2)^n$. By $\sum_{n=0}^{\infty}(1/2)^n$, we mean the sequence of numbers

$$\begin{aligned}
s_0 &= (1/2)^0 = 1 \\
s_1 &= (1/2)^0 + (1/2)^1 = 3/2 \\
s_2 &= (1/2)^0 + (1/2)^1 + (1/2)^2 = 7/4 \\
s_3 &= (1/2)^0 + (1/2)^1 + (1/2)^2 + (1/2)^3 = 15/8 \\
&\vdots
\end{aligned}$$

We will say in general a series CONVERGES if the sequence of partial sums s_n approach some limiting value, and we'll say the series DIVERGES otherwise. By the discussion of the geometric series at the beginning of this section,

$$s_n = \frac{1 - (1/2)^{n+1}}{(1 - 1/2)} = \text{ by some algebra } 2 - (1/2)^n.$$

So the partial sums s_n converge as n goes to infinity.

Another example: if $c_n = 1/n$, then by $\sum_{n=1}^{\infty} 1/n$ we mean the sequence of numbers

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
&\vdots
\end{aligned}$$

This example is a famous series, called the HARMONIC SERIES. Notice we started summing from $n = 1$ instead of $n = 0$; this is legal. The fact that this series diverges is discussed in exercise 6.4.

One more example: Take the Taylor series (4) for e^x at 0, and plug in $x = 1$. By $\sum_{n=0}^{\infty} 1/n!$ we mean the sequence

$$\begin{aligned}
s_0 &= 1/0! = 1 \\
s_1 &= 1/0! + 1/1! = 2 \\
s_2 &= 1/0! + 1/1! + 1/2! = 5/2 = 2.5 \\
s_2 &= 1/0! + 1/1! + 1/2! + 1/3! = 8/3 = 2.6666\dots \\
&\vdots
\end{aligned}$$

Later we'll see that this sequence really does converge to

$$e^1 = e = 2.71828182846\dots$$

WARNING: Try not to confuse power series, (which are formal objects with a variable x) and infinite series (which are actually sequences of real numbers, the partial sums.)

We will also need a more delicate notion of convergence. It sometimes happens that an infinite series will converge, but only because positive and negative terms cause a lot of cancellation. (To see an example, write down the first few partial sums of $\sum_n (-1)^n/n$.) For technical reasons, this is not optimal. So we'll also consider a new series formed by taking *absolute values*, and say that the series $\sum_n c_n$ CONVERGES ABSOLUTELY is the series of absolute values $\sum_n |c_n|$ converges, i.e. if the sequence

$$\begin{aligned} s_0 &= |c_0| \\ s_1 &= |c_0| + |c_1| \\ s_2 &= |c_0| + |c_1| + |c_2| \\ &\vdots \end{aligned}$$

converges. The point here is that the numbers being added are all positive, so the partial sums can only increase. There is never any cancellation. Absolute convergence is the very best possible, grade AAA kind of convergence. In particular, if a series converges absolutely, then it converges.

6.1. Write the first 5 partial sums for the series for the harmonic series. Do the same for the series $\sum_n (-1)^n/n$.

6.2. Write down the first 5 partial sums for the series $\sum_{n=1}^{\infty} 1/n^2$.

6.3. Use your calculator to compute the partial sum of the first 100 terms of the series in the previous problem. In fact this series converges; the Swiss mathematician Euler proved the very beautiful formula $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. This series is related to the function $L(x)$ in exercise 5.6. See also exercise 6.7.

6.4. In this exercise we show that the harmonic series diverges.

View each coefficient $1/n$ being summed as the area of a rectangle with base the interval $(n, n+1)$ of width 1 on the x axis, and height $1/n$. Draw a rough diagram of the first few rectangles. On this same graph, sketch the function $y = 1/x$. It should lie *under* all the rectangles. Thus the area under $y = 1/x$, from 1 up to some big integer N, is less than the sum of the areas of the rectangles, which is the N-th partial sum of the series $s_N = 1 + 1/2 + 1/3 + \cdots + 1/N$. Use calculus to compute the area under $y = 1/x$ from $x = 1$ up to $x = N$. Use this to show that as $N \to \infty$, $s_N \to \infty$.

This trick can often be made to work, its called the INTEGRAL TEST for convergence.

6.5. (Continuation) The antiderivative for $1/x$ you found above goes to infinity *very* slowly. How big does N have to be so that the area under $1/x$ between 1 and N is bigger than, say, 10? This tells you how many terms of the harmonic series you need before you can say that s_N is bigger than 10. The harmonic series diverges, but very slowly.

6.6. Try to generalize the arguments in exercise 6.4 to work for the series $\sum_{n=2}^{\infty} 1/(n \ln(n))$. You will need to integrate the function $y = 1/(x \ln(x))$.

6.7. The integral test can equally well be made to show a series converges. In this exercise you will show show that $\sum_{n=1}^{\infty} 1/n^2$ converges. First, what function $y = f(x)$ will be appropriate? Next, you need to arrange the rectangles so the graph of $y = f(x)$ lies *above* them instead of below. Compute

$$A_N = \int_1^N f(x)dx;$$

it is then *greater* than s_N. Show that $\lim_{N\to\infty} A_N$ is finite, then $\lim_{N\to\infty} s_N$ is finite as well, i.e. the series converges.

7. TESTS FOR CONVERGENCE

An application of the idea of absolute convergence is the COMPARISON TEST. It uses information about one series to get information about another. Here's how it works. Suppose $\sum_n c_n$ is some series that we know converges absolutely. Given another series $\sum_n b_n$, if $|b_n| \le |c_n|$ for all n, then the comparison test says that the series $\sum_n b_n$ also converges absolutely. The reason this works is that if $\sum |c_n|$ converges, its partial sums don't increase to infinity. The partial sums for $\sum_n |b_n|$ are all less than the partial sums for $\sum |c_n|$.

Conversely if the series with the *smaller* terms $\sum b_n$ does *not* converge absolutely, then neither does the series with the larger terms $\sum c_n$.

Amazingly, we can use this fact and our knowledge of the geometric series to get information about any power series at all. This is important enough to state in fancy language, but the ideas are simple enough for you to follow:

Theorem. *If $\sum_{n=0}^{\infty} c_n$ is any series, let L be the limit*

$$L = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}.$$

Then the series converges absolutely if $L < 1$, and it diverges if $L > 1$. There is no information if $L = 1$, anything can happen.

Proof. Suppose L as in the theorem satisfies $L < 1$; i.e

$$L = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1.$$

Since L is less than 1, it is a property of real numbers that there is another number r between L and 1. (For example, $r = (L+1)/2$ works.)

Since r is bigger than the limit L, its true that

$$\frac{|c_{n+1}|}{|c_n|} < r$$

for n big enough, say bigger than some N. (N depends on r, but is then fixed independent of everything else.) This says for example that

$$\frac{|c_{N+1}|}{|c_N|} < r \quad \text{or} \quad |c_{N+1}| < r|c_N|.$$

Similarly

$$\frac{|c_{N+2}|}{|c_{N+1}|}\frac{|c_{N+1}|}{|c_N|} < r^2 \quad \text{or} \quad |c_{N+2}| < r^2|c_N|.$$

In general we find that for any k

$$|c_{N+k}| < r^k|c_N|.$$

Since $r < 1$, the geometric series $|c_N|\sum_{k=0}^{\infty} r^k$ converges absolutely. By the comparison test, $\sum_{k=0}^{\infty}|c_{N+k}| = \sum_{n=N}^{\infty}|c_n|$ converges, i.e. the original series converges absolutely.

In the case when $L > 1$, one can show similarly that the individual terms $|c_n|$ are increasing, so the series diverges. □

We typically use this with $c_n = a_n(x-a)^n$ coming from some power series $\sum_n a_n(x-a)^n$ centered at $x = a$. The RADIUS OF CONVERGENCE R is the 'largest' value of x that makes the limit L equal 1. It is the radius of a one dimensional circle, the interval $(a-R, a+R)$. The ratio test then can be interpreted as saying that the series converges absolutely when $|x-a| < R$. We get no information when $|x-a| = R$. (The terminology 'radius' makes more sense when this is generalized to functions of two variables, then the power series converges inside a real circle in the plane.) The ratio test is extremely useful because so many power series involve $n!$, and the ratios cancel so nicely.

EXAMPLES: In the series for e^x, we have

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \frac{|x|^{n+1}}{|x|^n}\frac{n!}{(n+1)!} = \frac{|x|}{n+1}.$$

So the limit $L = 0$ for any x and therefore the series always converges absolutely, for any x. We say the radius $R = \infty$.

For the geometric series $a_n = 1$ for every n, so

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|x^{n+1}|}{|x^n|} = |x|.$$

Thus the the series converges absolutely if $|x| < 1$, as we already knew. The radius of convergence is 1

The series for $\ln(1-x)$ is $\sum_{n=1}^{\infty} -x^n/n$, so

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|x|^{n+1}/(n+1)}{|x|^n/n} = |x|\frac{n}{(n+1)} = |x|\frac{1}{1+1/n}.$$

The last step above comes from multiplying numerator and denominator by $1/n$, useful for seeing that the limit is $|x| \cdot 1 = |x|$. Thus the series converges absolutely for $|x| < 1$, or $R = 1$.

Here's a slightly more complicated example; the power series

$$\sum_{n=0}^{\infty} x^{3n}/2^n.$$

(See exercise 4.3). We have

$$\frac{|x|^{3(n+1)}/2^{n+1}}{|x^{3n}|/2^n} = |x|^3/2.$$

Thus the series converges absolutely if $|x|^3 < 2$, or, $|x| < 2^{1/3}$. This show the radius of convergence is $R = 2^{1/3}$.

7.1. Use the comparison test and the harmonic series to show that $\sum_n 1/\sqrt{n}$ diverges. (You may assume the harmonic series diverges, even if you didn't do exercise 6.4.) Hint: $\sqrt{n} < n$; what does this say about $1/\sqrt{n}$ vs. $1/n$?

7.2. Use the comparison test and your answer to exercise 6.7 to show that the series $\sum_{n=1}^{\infty} 1/n^3$ converges.

7.3. Use the ratio test to find the radius of convergence for $\sinh(x)$ (see exercise 4.4) and for $\sin(x)$.

7.4. Find the radius of convergence of the Taylor series for $x^2 \cos(3x)$ you computed in exercise 4.2

7.5. Fin the the radius of convergence of the Taylor series for $\sin(x)/x$ and $(e^x - 1)/x$ you computed in exercise 4.5

7.6. Show that the radius of convergence of $\sum_{n=1}^{\infty} n!x^n$ is 0. This shows the worst case can happen.

7.7. Compute the radius of convergence for the Bessel function $J_0(x)$ given by formula (8). This shows that $J_0(x)$ really does define a function. Use the first 4 terms and your calculator to estimate $J_0(1)$.

7.8. Use the ratio test to show that $R = 1$ for the series $L(x)$ you found in exercise 5.6. Notice that exercise 6.7 shows that the series $L(1)$ *does* converge, even though this is on the boundary, where the ratio test gives no information. The geometric series also has radius $R = 1$, but certainly does not converge when $x = 1$. (What is the Nth partial sum for the geometric series with $x = 1$?) This shows anything can happen on the boundary.

7.9. Determine the radius of convergence of the binomial series (exercise 4.6). There are different answers depending on whether or not t is an integer.