CHAPTER I
FUNCTIONS OF TWO VARIABLES

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ADVICE TO THE INSTRUCTOR: This chapter should precede the material on ODEs for Math 3C. (The functions we examine in the exercises here were chosen because they reappear in the exercises to section 1.2 of the Math 3C text. Try to make use of this when you get to that section.) There is very little material here, barely enough for a week. On the other hand, Chapter II contains more than enough for two weeks.

1. EXAMPLE OF FUNCTIONS OF TWO VARIABLES

So far in calculus you have studied functions of one variable. This is the starting point because it is the easiest case. In practical applications, however, things usually depend on several variables. In this chapter we consider two variables instead of just one.

For example, imagine waves crashing against the breakwater down at the Santa Barbara yacht basin. The height of the water depends on both the distance x along the breakwater (different heights at different places along the breakwater), and also on the time t (different heights at different times). Neither variable controls the height by itself. In economics, it is often the case that the cost of some commodity depends on both the time it is bought (strawberries cost more in November), and the quantity being purchased (price per pound is less if you buy in bulk.) The monthly payment on your car loan depends on the interest rate, as well as on the amount you borrowed.

Many functions of two variables can be given by simple formulas involving the variables. We might model waves with a function like $h(x,t) = \sin(x+t)$. A fluctuation in cost might be described by $c(x,t) = \sin(t)/x$, where t is still time and x is quantity. Another example: in physics, Boyle's law says that the temperature T of a gas is proportional to both the volume V of the gas and the pressure P the gas is under:

$$T = \text{a constant} \times PV$$

These are fairly simple ideas; we haven't said anything deep here.

2. Graphs in three-space

The graph of a function of a single variable $y = f(x)$ is a set of points in the plane, the points (x, y) where $y = f(x)$. For example, if $y = x^2$, the graph is exactly those points of the form (x, x^2). As you learn in precalculus, the graph of a function is some sort of curve in the plane that passes the 'vertical line test': any vertical line in the plane crosses it at most once.

Thus if we have instead two input variables now together with one output variable, the graph consists of points in three dimensional space. It is some kind of surface. It is harder to visualize, and we can't represent it exactly on a two dimensional piece of paper. We can only show projections of it. This section and the next will give you some tools for visualizing graphs of functions of two variables.

Suppose we call the input variables x and t. Imagine the floor covered with Linoleum floor tiles, making a grid. Pick some point to be the origin, and imagine x being measured on the floor in one direction, and t measured at right angles. The floor is the two dimensional plane where the input variables live. Now we measure another variable y vertically, distance above the floor. For each point (x, t) in the plane, the function determines a number $f(x, t)$ which we interpret as a height. The point $(x, t, y = f(x, t))$ in three space is a point on the graph. The set of all such points as x and t vary determine the graph, which looks like some kind of surface.

The simplest way to get a handle on what the surface looks like is to just go back to the case of one variable. For example, consider the function for the wave breaking

$$h(x, t) = \sin(x + t)$$

We can fix an x value $x = x_0$ some number and think about what happens as t varies. In real world terms, we pick some point on the breakwater and measure height at that fixed point, as a function of time. Now we have a function of only one variable, t:

$$h(x_0, t) = \sin(x_0 + t).$$

We understand each of these graphs; they are $\sin$ curves, shifted by the amount x_0. We show four of them, for $x_0 = 0, \pi/4, \pi/2, 3\pi/4$ in Figure 1 below. Geometrically what we have done by fixing $x = x_0$ is to draw a line (on the floor, perpendicular to the x axis at x_0) and cut the surface above that line. We then imagine a piece of paper placed vertically in that cut. The cut edge traces out a curve on the paper: one of the graphs in Figure 1. We have found what a CROSS-SECTION

Figure 1

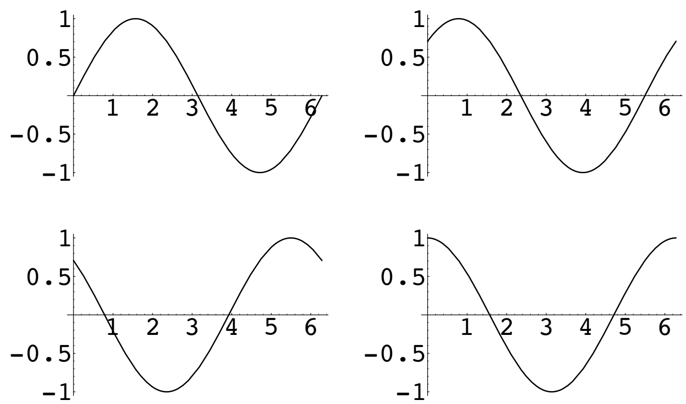
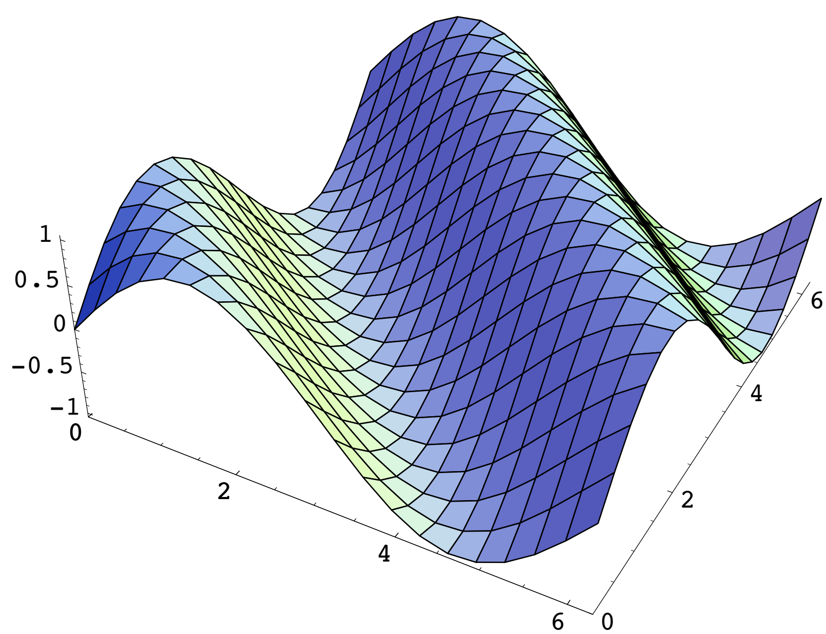


Figure 2



of the graph looks like. Figure 2 shows a computer generated graph of $y = \sin(x + t)$.

We can also fix a time $t = t_0$ and graph $\sin(x+t_0)$ as a function of x. (Of course, we then see a similar picture.) In terms of the graph, this is a cut along a line perpendicular to the t axis. In real world terms,

Figure 3

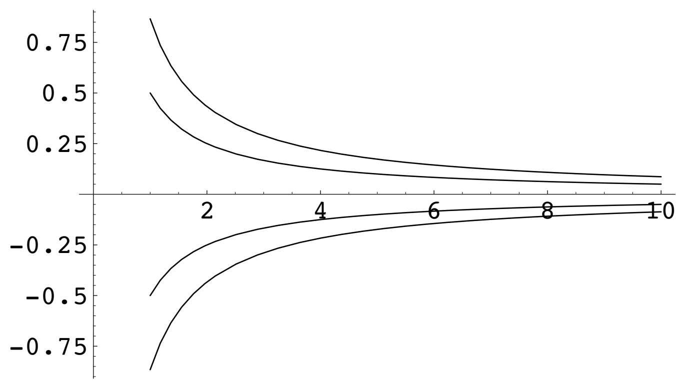
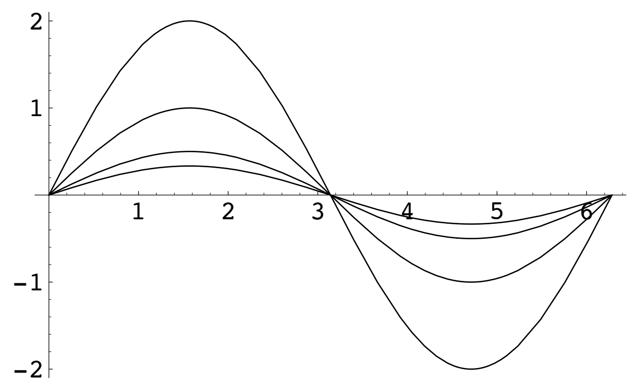


Figure 4



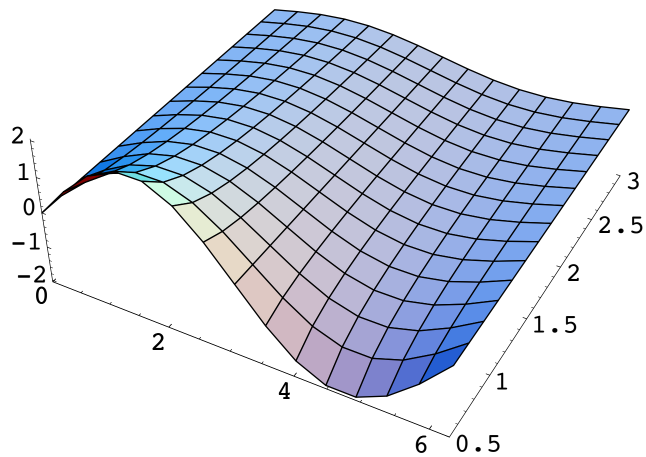
this is freezing the wave at some instant in time, and graphing the height all along the breakwater at that time.

For another example, consider the function

$$c(x,t) = \sin(t)/x$$

We are still measuring t and x on the plane, and c is a vertical distance above the point (x,t). If we fix a time t_0, then $\sin(t_0)$ is a number, and we understand the graph of $c(t_0,x) = \sin(t_0)/x$: it is a hyperbola (like $1/x$) scaled by the number $\sin(t_0)$. Figure 3 shows several, for $t_0 = -\pi/3, -\pi/6, \pi/6, \pi/3$. On the other hand, a cross section in the other direction with $x = x_0$ fixed gives a $\sin$ curve, with 'amplitude' $1/x_0$. Figure 4 shows the cases $x_0 = 1/2, 1, 2, 3$. Figure 5 shows a computer generated graph of $c(x,t) = \sin(t)/x$.

Figure 5



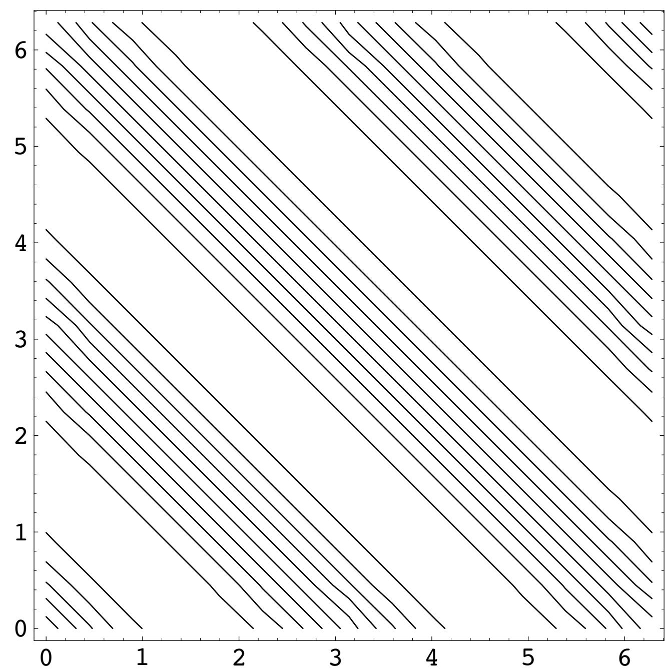
2.1. Sketch several cross sections for each of the functions $f(t, y) = 2t - y$, $f(t, y) = -ty$, and $f(t, y) = y^2 - t$. Save these results, they will come in useful in section 1.2 of the Math 3C text. (For consistency with that section I've used t for one of the dependent variables instead of x.)

3. LEVEL CURVES

In finding cross sections above, we just ignored one or the other of the input variables. In this section, we ignore the *output* variable. Amazingly, this turns out to be helpful. A geometric explanation makes it clear. In three space with coordinates (x, t, y) there are some two-dimensional planes. The 'Linoleum floor' mentioned above is all points of the form $(x, t, 0)$. If we fix any other y value $y = y_0$ we get a plane parallel to it above (or below) distance y_0. Analogous to the cross sections we formed in section 2 we can slice through the graph with a plane parallel to the floor. This determines some curve in the plane. This is called a LEVEL CURVE.

Algebraically, we fix a number y_0 and solve for all points (x, t) in the plane such that $f(x, t) = y_0$ The difference is that level curve may not be the graph of a function. For example, if $f(x, t) = x^2 + t^2$, then all the cross sections are parabolas (see exercise 3.2.) But what do the level curves look like? Fix $y = 1$, we are looking for the set of points (x, t) in the plane such that $x^2 + t^2 = 1$, which we recognize is a circle. A perfectly nice set of points, but not the graph of a function $x = x(t)$

Figure 6



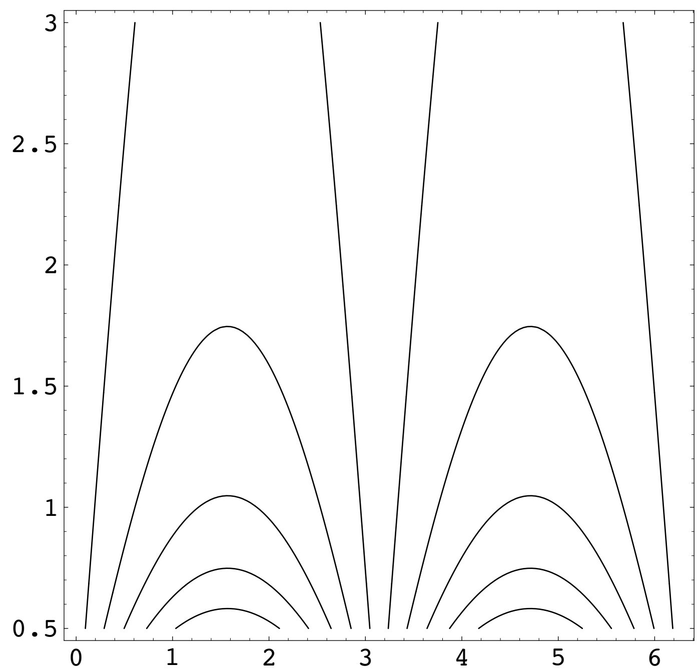
or $t = t(x)$. All the level curves will be some kind of circle, although the circle of radius 0 (the point $(0, 0)$ corresponding to $y_0 = 0$) and the empty circle (corresponding to $y_0 < 0$) are possibilities.

For another example, consider the function $h(x, t) = \sin(x + t)$. Clearly there are no points on the graph with height > 1 or < -1. If we fix any other height y_0, with $-1 \leq y_0 \leq 1$, then $\sin(x + t) = y_0$ exactly when $x + t = \arcsin(y_0)$, some other constant. So $x = -t + \arcsin(y_0)$. The level curves form lines in the plane, all with slope -1. Note that the periodicity of $\sin$ means $\arcsin$ is multiply valued. For example $\arcsin(0) = 0, \pm\pi, \pm 2\pi, \ldots$ A set of level curves in the plane is often called a CONTOUR DIAGRAM; see Figure 6. Compare it to the graph of the function in Figure 2.

For the function $c(x, t) = \sin(t)/x$ from section 2, fix a height c_0 above the (x, t) plane and set $c_0 = \sin(t)/x$. We get $x = \sin(t)/c_0$. Thus x looks like a function of t in this case, a $\sin$ curve with amplitude $1/c_0$. Figure 7 shows a computer generated contour diagram. Compare Figure 7 with Figure 5.

Actually we've missed a case, and the computer failed to show it, too. If $c_0 = 0$ we can't divide by it. This is the level curve at height 0, where the graph crosses the (x, t) plane. We see instead

Figure 7



that $\sin(t)/x = 0$ exactly when $\sin(t) = 0$, i.e. $t = 0, \pm\pi, \pm 2\pi, \ldots$ These are vertical lines in the plane (if we think of t as being on the 'horizontal' axis), so they are not of the form x as a function of t like the others.

3.1. Sketch several level curves for each of the functions considered in exercise 2.1

3.2. Consider the function $g(x, t) = \sqrt{x^2 + t^2}$. What do the level curves look like? How do they compare to the level curves for the function $f(x, t) = x^2 + t^2$ considered in the example above? How do the graphs of the two functions compare? Use the method of cross sections from section 2 to find out.

4. Partial derivatives

The point of derivatives in the calculus of functions of one variable is to approximate the graph of the function by a straight line. This section generalizes that idea to functions of two variables.

Actually, we still see functions of one variable when we form the cross sections. These graphs each have a tangent line at any point. View the cross section as we did before as a cut along the graph in three space. Then our tangent line is a line in three dimensional space, perpendicular to one of the axes, and tangent to the cross section of the graph. There are two ways to do this depending on which variable we fix.

When we do this, the slope of that line is what we call a PARTIAL DERIVATIVE. The notation for these is a little more complicated now because we have to keep track of the function and which variable is fixed or changing. If we have $y = f(x,t)$ we write $\frac{\partial f}{\partial x}$ to mean the slope of the tangent line when x is varying and t is fixed. Other shorthand ways to write this are f_x or even y_x. If we instead fix x and vary t, the slope of the tangent line is written $\frac{\partial f}{\partial t}$ (or f_t, or y_t.) Sometimes the function doesn't have a name of its own, its just an expression like $\sin(x+t)$; we write

$$\frac{\partial}{\partial x}\left(\sin(x+t)\right) \quad \text{or} \quad \frac{\partial}{\partial t}\left(\sin(x+t)\right)$$

to mean the corresponding partial derivatives.

How do we compute such things? Its easy; all the rules of Math 3A still apply to $f(x,t)$ when we compute f_x. We just need to remember that t is *fixed*, it is a *number, not a variable*. For example, with $c(x,t) = \sin(t)/x$, we see that $c_x = -\sin(t)/x^2$, since in Math 3A, the derivative of $1/x$ is $-1/x^2$, and similarly for the derivative of any number times $1/x$. Meanwhile $c_t = \cos(t)/x$ since the derivative of $\sin(t)$ is $\cos(t)$. Of course, we don't get a number for the slope until we plug in a number for the fixed variable (which cross section are we looking at?) as well as a number for the other variable (where along the cross section do we want the tangent line?) For example, $c_t(0,2)$ means plug in $t = 0$ and $x = 2$; $c_t(0,2) = \cos(0)/2 = 1/2$. We might also write

$$\frac{\partial c}{\partial t}\Big|_{(0,2)} = \frac{1}{2}$$

for this.

You will probably need to see some more examples. For $h(x,t) = \sin(x+t)$ we see that

$$h_x = \cos(x+t), \quad \text{and} \quad h_t = \cos(x+t) \quad \text{also.}$$

For the function $g(x,t) = \sqrt{x^2+t^2}$ of exercise 3.2, we find that

$$g_x = \frac{1}{2}(x^2+t^2)^{-1/2} \cdot 2x$$

by the chain rule, since the derivative with respect to x of $x^2 + t^2$ is $2x$ (remember, here t is a number.) But

$$g_t = \frac{1}{2}(x^2 + t^2)^{-1/2} \cdot 2t$$

again by the chain rule, since the derivative with respect to t is $2t$ (now x is just a number.)

The exercises in this section should be very easy. Just work slowly and keep reminding yourself which variable is fixed. In each, find the indicated partial derivative.

4.1. F_v and F_r if $F = mv^2/r$

4.2. $\frac{\partial}{\partial m}\left(\frac{1}{2}mv^2\right)$, $\frac{\partial}{\partial v}\left(\frac{1}{2}mv^2\right)$

4.3. f_x and f_y for $f(x, y) = \sin(x - y)$.

4.4. z_x for $z = x^2y + 2xy^3$

4.5. $\frac{\partial}{\partial r}\left(\frac{2\pi r}{v}\right)$, $\frac{\partial}{\partial v}\left(\frac{2\pi r}{v}\right)$

4.6. z_y if $z = xe^{-y} + ye^x$

4.7. $\partial T/\partial V$ if $T(P, V) = 2PV$

4.8. T as above, what is $T_V(0, 3)$?

4.9. $f(x, y) = x^3y^5$, compute $f_x(3, -1)$

4.10. $h(x, y) = \sqrt{2x + 3y}$, find $h_y(2, 4)$

4.11. $\frac{\partial}{\partial x}\left(x \ln(y \cos(x))\right)|_{(\pi/3,1)}$

5. Tangent planes

In the calculus of functions of one variable, we use the *slope* of the tangent line to deduce the *equation* of the tangent line. This section generalizes that idea to functions of two variables.

The analog of a line tangent to a curve should be a plane tangent to a surface. How is a plane in three-space described by an equation? The plane has the property that any cross section of it is a line; so the equation of a plane should have an analogous property: if we plug in a number for one variable, we get an equation for a line in the other variable. One way to write this is

$$y = Ax + Bt + C$$

for some constants A, B, and C. Suppose we want the plane to pass through a particular point in three space (x_0, t_0, y_0), where $y_0 =$

$f(x_0, t_0)$ is a point on the graph. Then this point must be a solution to the equation, so we can re-write it

$$y = A(x - x_0) + B(t - t_0) + y_0.$$

(In other words, saying the plane passes through a particular point determines one of the parameters.) Suppose we also want the plane to be tangent to the graph. Then certainly the cross section tangent lines must match up, and in particular their slopes must match. This means

$$\frac{\partial}{\partial x}\left(A(x - x_0) + B(t - t_0) + y_0\right)|_{(x_0,t_0)} = \frac{\partial f}{\partial x}(x_0, t_0)$$

or $A = f_x(x_0, t_0)$. Similarly $B = f_t(x_0, t_0)$. So the equation of the plane tangent to the graph at the point $(x_0, t_0, f(x_0, t_0))$ is given by

$$y = f(x_0, t_0) + f_x(x_0, t_0)(x - x_0) + f_t(x_0, t_0)(t - t_0).$$

For example, with the function $\sin(t)/x$ at the point $(2, \pi/3)$ we compute

$$\begin{aligned} \sin(\pi/3)/2 &= \sqrt{3}/4 \\ \frac{\partial}{\partial x}\left(\sin(t)/x\right)|_{(2,\pi/3)} &= -\sin(t)/x^2|_{(2,\pi/3)} = -\sqrt{3}/8 \\ \frac{\partial}{\partial t}\left(\sin(t)/x\right)|_{(2,\pi/3)} &= \cos(t)/x|_{(2,\pi/3)} = 1/4 \end{aligned}$$

So the equation of the tangent plane at $(2, \pi/3)$ is

$$y = \sqrt{3}/4 - \sqrt{3}/8(x - 2) + 1/4(t - \pi/3).$$

This equation has the property that it *approximates* the original function near the point:

$$\sin(t)/x \approx \sqrt{3}/4 - \sqrt{3}/8(x - 2) + 1/4(t - \pi/3)$$

for (x, t) near to $(2, \pi/3)$.

5.1. Compute the equation of the tangent plane for the function and point specified in exercise 4.9

5.2. Compute the equation of the tangent plane for the function and point specified in exercise 4.10

5.3. Compute the equation of the tangent plane for the function and point specified in exercise 4.11