Math 303 Exam 1 Solutions

Feb 27, 2008

1. (10 pts) Identify the largest number expressible in a *simple grouping system* which uses a base of 5 and 10 distinct symbols. Give your answer as a number in the Hindu-Arabic decimal system. Explain carefully why this is the largest possible number.

A simple grouping system is like the traditional Egyptian number system. It uses symbols to represent powers of the base. With 10 distinct symbols, we can represent the first ten powers of the base: $5^0, 5^1, 5^2, \ldots, 5^9$. We can combine up to 4 of each symbol to make up a number. So the largest number we can represent is

$$\begin{aligned} 4 \cdot 5^9 + 4 \cdot 5^8 + \cdots + 4 \cdot 5 + 4 &= 4(5^9 + 5^8 + \cdots + 5 + 1) \\ &= 4\,\frac{5^{10} - 1}{5 - 1} = 5^{10} - 1 = 9765624 \end{aligned}$$

Another way to look at this is that a simple grouping system with 10 distinct symbols has the same ability to describe numbers as a place-value system with 10 digits. The largest number that can be described with 10 digits base 5 is

$$4444444444_5 = 10000000000_5 - 1_5 = 5^{10} - 1 = 9765624.$$

2. (10 pts) Convert the numbers below to Egyptian numerals and compute the product using the Egyptian algorithm. Use Egyptian numerals in the steps of your computation.

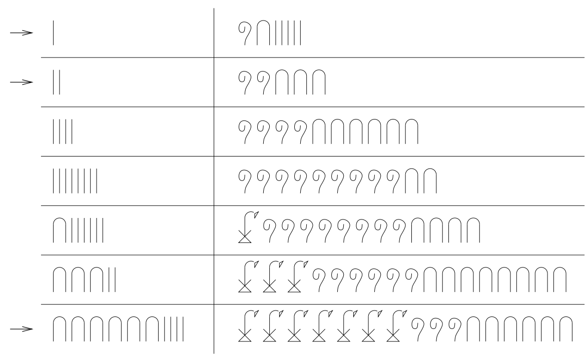
$$67 \times 115$$

Here is a table of Egyptian numerals:

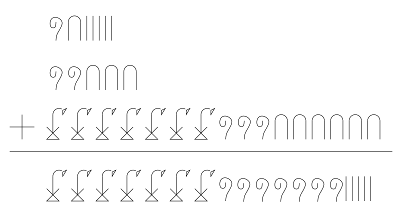
TABLE 1 Early Egyptian Symbols

Number	Symbol	Description
1	𓏺	Stroke
10	𓎆	Heel bone
100	𓍢	Scroll
1000	𓆼	Lotus flower
10,000	𓂭	Pointing finger
100,000	𓆐	Burbot fish
1,000,000	𓁨	Astonished person

First the multiplication:



Now the addition:



3. (10 pts) Consider the following table:

J	m	n	p
m	n	p	n
n	p	m	n
p	n	n	m

(a) Is J an operation on the set $S = \{m, n, p\}$ (i.e. is S closed under J)? Why?

Yes, it is. Every entry in the body of the table is in S, so S is closed under J. That is, J is an operation on S.

(b) Is J commutative? Why?

The table is symmetric with respect to the diagonal from the upper left to the lower right corner. This shows that $xJy = yJx$ for any x, and y in S. Hence J is commutative.

(c) Does J have an identity? Why?

No, it does not. If there were an identity then the row and column corresponding to that element would have to be copies of the header row and column of the table. None of the rows (or columns) is a copy of the header row (or column), hence there is no identity.

(d) Does each element have an inverse? Why?

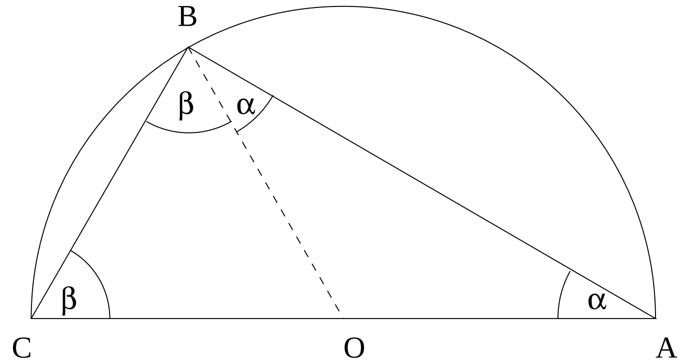
Since there is no identity, no element can have an inverse.

4. (10 pts)

(a) State and prove Thales' Theorem.

Thales' Theorem: An angle inscribed in a semicircle is a right angle.

Proof: Let $\angle ABC$ be such an angle as in the figure below.

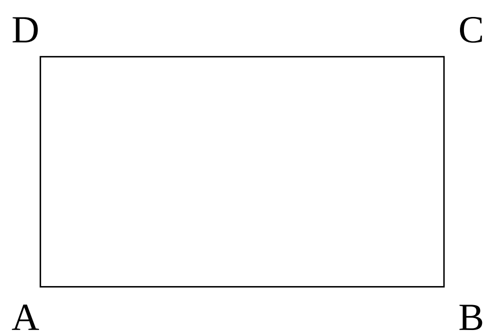


Add the line segment OB from the center of the semicircle to the vertex of the angle. Now consider the triangle ABO. The two sides OA and OB are both radii of the semicircle, hence they are equal. Therefore the opposite angles denoted by α must be equal too. Similarly, OBC is an isosceles triangle and the angles denoted by β are equal. Therefore the sum of the angles of the triangle ABC is $2\alpha + 2\beta = 2(\alpha + \beta)$, which must be two right angles. This shows that $\alpha + \beta$ is a right angle.

(b) When and where did Thales live? (A century and a country-size region of the world will suffice. You don't need to know the exact years and the town, although if you do, I'll be impressed.)

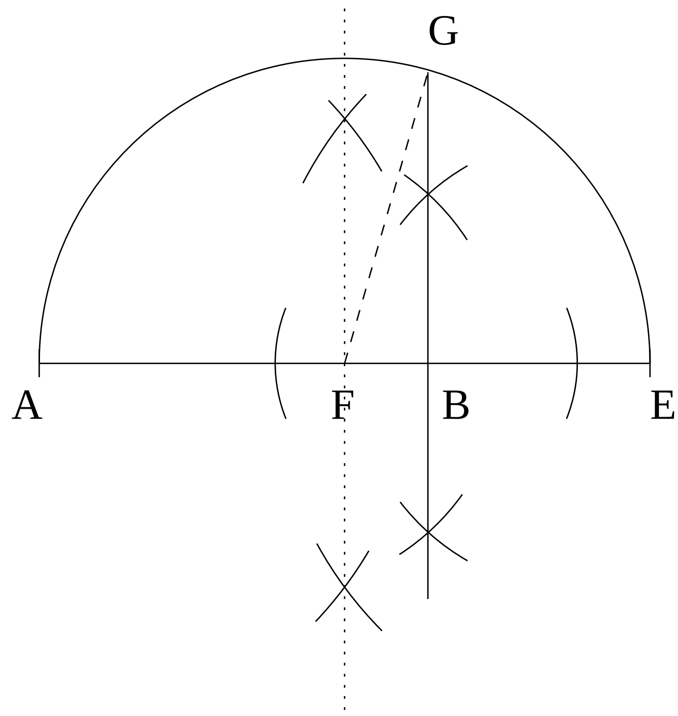
He lived in Miletus in Asia Minor around 640–546 BC (estimates vary somewhat). This was part of ancient Greece at the time. It is presently in Turkey.

5. (10 pts) Given the rectangle below, construct a square of the same area using only straightedge and compass. Explain what you do at each step of the construction. You may use standard steps of geometric construction without proving why they work.



1. Extend side AB past B and copy BC on there so that $BE = BC$.
2. Open the compass to a little more than half of the line segment AE and make arcs centered at A and E. These arcs will intersect in two points. Connect those two points with a line. The resulting line intersects AE at its midpoint. Call this point F.
3. Draw a semicircle centered at F with AE as diameter.
4. Construct a perpendicular to AE at the point B. You can do this by marking off points equidistant from B on both sides of it with the compass then using the same construction with these two points as in 2 to draw a perpendicular which will go through B. Call the intersection of this perpendicular with the semicircle G.

5. Construct a square with sides equal to BG. This can be done by copying the segment BG with the compass and constructing perpendiculars as we did in 4.



We need to justify why this procedure gives a square with the same area as the rectangle $ABCD$. The area of the rectangle is

$$\begin{aligned}\overline{AB}\,\overline{BC} &= \overline{AB}\,\overline{BE} = (\overline{AF}+\overline{FB})(\overline{EF}-\overline{FB}) \\ &= (\overline{GF}+\overline{FB})(\overline{GF}-\overline{FB}) && \text{since } \overline{EF}=\overline{AF}=\overline{GF} \\ &= \overline{GF}^2 - \overline{FB}^2\end{aligned}$$

Since the triangle FBG has a right angle by contruction, the Pythagorean Theorem shows

$$\overline{GF}^2 - \overline{FB}^2 = \overline{BG}^2$$

which is exactly the area of the square we constructed.

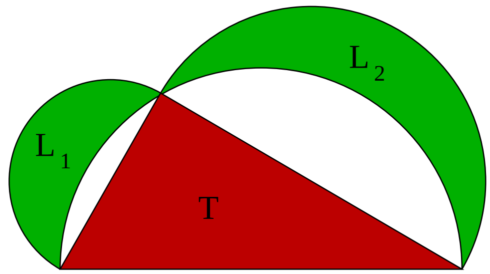
6. (10 pts)
   (a) Explain what it means for two line segments to be commensurable.

   Two line segments are commensurable if there exists a smaller line segment such that both original segments are integer multiples of this smaller segment.

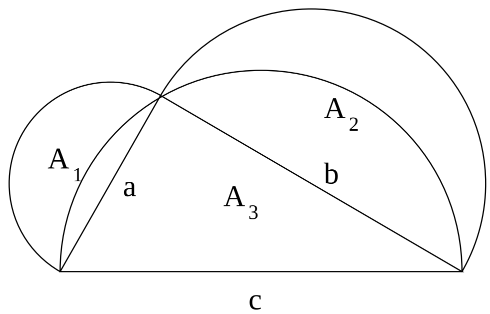
   (b) Hippasus was the first to notice that not all pairs of line segments are commensurable. In particular, what two line segments did he prove were not commensurable? (You don't need to give the proof, just state what the line segments are.)

   He proved that the side of a square and the diagonal of the same square were not commensurable.

7. (15 pts) **Extra credit problem.** Prove that the sum of the areas of the shaded lunes (L_1 and L_2) in the figure below is equal to the area of the shaded triangle (T).



The idea of this proof is the same as the argument we used in the quadrature of the lune. First, consider the figure below, where A_1, A_2 and A_3 are the areas of the three semicircles.



The ratios of the areas are the squares of the ratios the diameters. Hence

$$\frac{A_1}{A_3} = \left(\frac{a}{c}\right)^2 = \frac{a^2}{c^2} \implies A_1 = \frac{a^2}{c^2}A_3$$
$$\frac{A_2}{A_3} = \left(\frac{b}{c}\right)^2 = \frac{b^2}{c^2} \implies A_2 = \frac{b^2}{c^2}A_3$$

Therefore

$$A_1 + A_2 = \left(\frac{a^2}{c^2} + \frac{b^2}{c^2}\right) A_3 = \frac{a^2 + b^2}{c^2} A_3.$$

By Thales' Theorem, the triangle has a right angle, hence by the Pythagorean Theorem, $a^2 + b^2 = c^2$. This shows $A_1 + A_2 = A_3$. Now, we get $L_1 + L_2$ by removing the two unshaded circular segments from $A_1 + A_2$. Likewise, we get T by removing the same two unshaded circular segments from A_3. Therefore $L_1 + L_2 = T$.

Note: This is a remarkable result. It shows that the areas of the two lunes combined can always be squared (since the area T can be squared) even though individually the lunes can almost never be squared.