Math 303 Exam 2 Solutions

Nov 4, 2009

1. (5 pts each)
   (a) Show that the sum of a rational and an irrational number is an irrational number.

   Let x be rational and y irrational. Then $z = x + y$ is certainly a real number. We will prove that z is irrational by contradiction. If z is rational then $z = m/n$ for some m, n integers, $n \neq 0$. Since x is rational, $x = p/q$ for some p, q integers, $q \neq 0$. Now

   $$\frac{m}{n} = \frac{p}{q} + y \implies y = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}.$$

   But this is a rational number, since $mp - qn$ and nq must be integers as m, n, p, q are all integers and $nq \neq 0$ as $n, q \neq 0$. This contradicts the fact that y is irrational. Therefore our assumption that z is rational must be false.

   (b) Prove that between any two distinct irrational numbers there are infinitely many irrational numbers.

   Let x and y be any two distinct irrational numbers. Without loss of generality, suppose $x < y$. So $y - x > 0$. However small $y - x$ is, if n is a sufficiently large positive integer $1/n < y - x$. Now consider the numbers

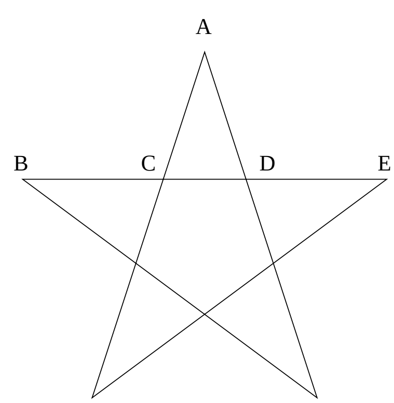
   $$x + \frac{1}{n}, x + \frac{1}{n+1}, x + \frac{1}{n+2}, \ldots$$

   Notice that all the numbers in the above infinite list are irrational because they are the sum of an irrational and a rational. They are also all between x and y because

   $$y - x > \frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2} > \cdots.$$

   Finally, they are also all different. Therefore this list contains infinitely many irrational numbers between x and y.

2. (10 pts) The symbol of the Pythagorean brotherhood was the *pentagram*, or five-pointed star formed by the five diagonals of a regular pentagon. Prove that each of the five sides of the pentagram divides in golden section the two sides of the pentagram that it intersects.



Notice that the pentagon in the middle of the pentagram is a regular pentagon, by symmetry of the pentagram. Hence its angles are $108°$. This makes the base angles of the isosceles triangles on the faces of this pentagon $72°$. So $\triangle ACD$ is a $72° - 72° - 36°$ triangle. We know that in such a triangle, AC/CD is the golden ratio, i.e. $AC/CD = (\sqrt{5}+1)/2$. This shows

$$AC = BC = DE = \frac{\sqrt{5}+1}{2} CD$$

Hence

$$BD = BC + CD = \left(\frac{\sqrt{5}+1}{2} + 1 \right) CD = \frac{\sqrt{5}+3}{2} CD.$$

and

$$\frac{BD}{DE} = \frac{\frac{\sqrt{5}+3}{2}CD}{\frac{\sqrt{5}+1}{2}CD} = \frac{\sqrt{5}+3}{\sqrt{5}+1}$$
$$= \frac{\sqrt{5}+3}{\sqrt{5}+1}\,\frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{(\sqrt{5}+3)(\sqrt{5}-1)}{5-1}$$
$$= \frac{5+2\sqrt{5}-3}{4} = \frac{5+2\sqrt{5}-3}{4} = \frac{\sqrt{5}+1}{2}.$$

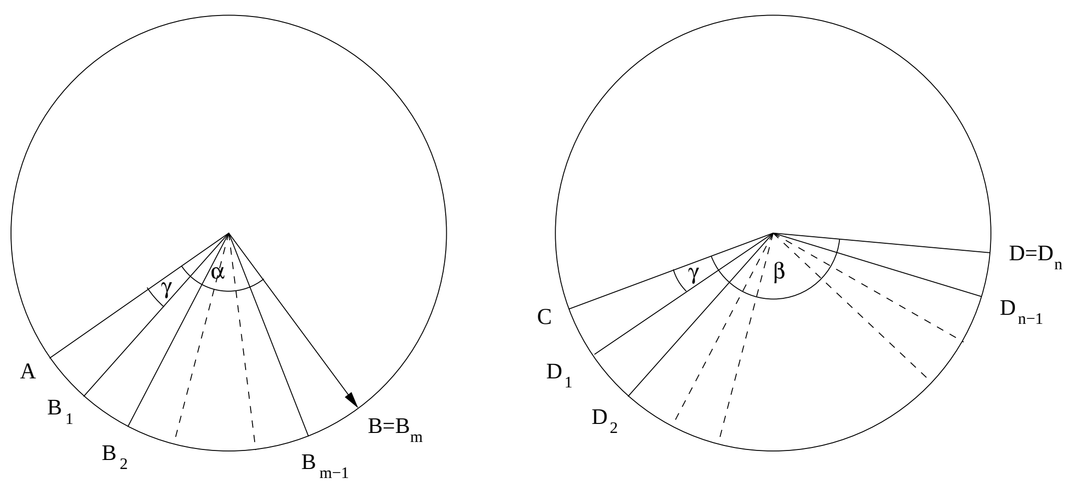
This shows BD/DE is the golden ratio. That all the other sides are divided in the same way follows from symmetry.

Alternately, if you prefer to think of the golden ratio as the mean proportional, see the solution on p. 233 of your textbook.

3. (10 pts) Consider the proposition:

Proposition. *Central angles in the same or equal circles are to each other as their intercepted arcs.*

Establish, in Pythagorean fashion, the case where the two central angles are commensurable.



Consider two circles of equal radius r with two central angles α and β. Suppose there is an angle γ such that $\alpha = m\gamma$ and $\beta = n\gamma$, where $m, n \in \mathbb{Z}^+$. Divide α into m equal parts and β into n equal parts. So each part is γ. This must divide the arc AB into m parts and the arc CD into n parts. All these small arcs have the same length because they are intercepted by the same central angle γ. So arc $AB = m(\text{arc } AB_1)$ and arc $CD = n(\text{arc } CD_1)$. But arc $AB_1 =$ arc CD_1 because they are intercepted by the same central angle in circles of the same radii. So

$$\frac{\alpha}{\beta} = \frac{m\gamma}{n\gamma} = \frac{m}{n} = \frac{m(\text{arc } AB_1)}{n(\text{arc } CD_1)} = \frac{\text{arc } AB}{\text{arc } CD}.$$

4. (a) (10 pts) Describe the system of material axiomatics.

Material axiomatics is a logical system for presenting mathematics. Its pattern is

(i) First certain undefined primitives are specified, possibly along with intuitive explanation of their meanings.

(ii) Further terminology is defined in terms of the undefined primitives and previously defined terms.

(iii) The axioms (or postulates) are stated without definition, although possibly along with some motivating explanation, to describe the properties of the undefined primitives.

(iv) Further theorems are stated and deduced from the axioms and previously proven theorems.

(b) (2 pts) When and where did material axiomatics develop?

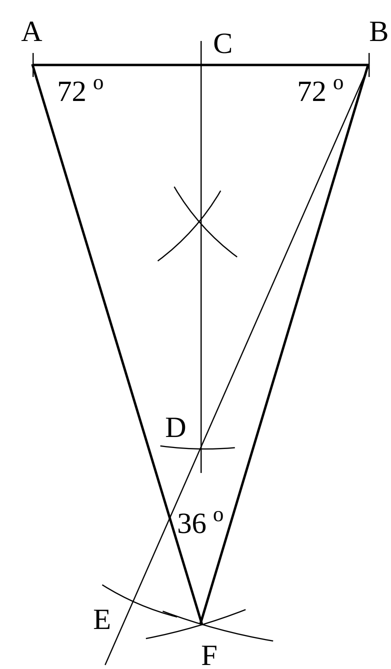
In Greece, sometime between 500-300 BC.

(c) (8 pts) Why was material axiomatics a major advance in the development of mathematics? Give at least one significant problem that material axiomatics resolved.

It allowed for a systematic presentation of mathematical knowledge from the clearly stated foundations to the more complex statements in a way that largely remains in use today. It solved a number of problems. For example,

- It avoided circularity of logic in definitions and proofs by separating primitives and axioms from definitions and theorems.
- It avoided the infinite descent problem in definitions and proofs by clearly stating what undefined terms could be used in defining other terms and what statements were accepted without proof.
- It allowed theorems to be traced back to the axioms that were used in their proofs, thereby making it possible to know what theorems remain valid if an axiom is changed or dropped later.

5. (10 pts) Using straight edge and compass, construct a $72° - 72° - 36°$ triangle whose base is given below. Be sure to give a step-by-step description of what you did.

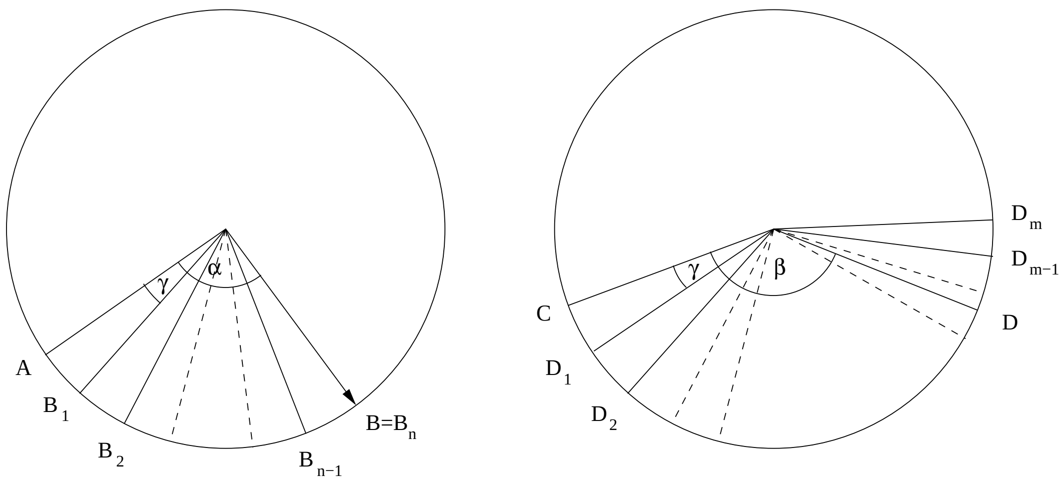


1. Construct a perpendicular bisector to AB by drawing circles of radius AB centered at A and B and connecting the two intersections of these circles.
2. Mark off D a distance of AB from C on this bisector.
3. Connect B and D. At this point, $BD = \frac{\sqrt{5}}{2}AB$ by the Pythagorean Theorem.
4. Extend BD from D and mark off E a distance of AC from D. Now $BE = \frac{\sqrt{5}+1}{2}AB$.
5. Draw circles of radius BE centered at A and B. Call one of the intersections of these circles F.
6. Connect F to A and B. Since $AF = BF = \frac{\sqrt{5}+1}{2}AB$, the $\triangle ABF$ is a $72° - 72° - 36°$ triangle with base AB.

6. **Extra credit problem.**

(a) (10 pts) Prove the proposition in Problem 3 in the general case (when the two central angles may not be commensurable) using the Eudoxian method of proportions.

I will assume that the following is already known: equal central angles intercept equal arcs in two circles of the same radii.



Let $m, n \in \mathbb{Z}^+$. Divide α into n equal parts. Notice that

$$\text{arc } AB_1 = \text{arc } B_1B_2 = \cdots = \text{arc } B_{n-1}B$$

because they are all intercepted by the same central angle in the same circle and then arc $AB = n(\text{arc } AB_1)$. Now measure m γ's from C. Notice that

$$\text{arc } AB_1 = \text{arc } CD_1 = \text{arc } D_1D_2 = \cdots = \text{arc } D_{m-1}D_m$$

because they are arcs intercepted by the same angle in the same or equal circles. Note that D_m does not coincide with D in general.

Now there are the usual three cases

- If $m\alpha < n\beta$ then

$$m\frac{\alpha}{n} < \beta \implies m\gamma < \beta.$$

  Hence the circular sector cut out by $m\gamma$ fits inside the circular sector cut out by β. It follows that

$$m(\text{arc } AB_1) < \text{arc } CD \implies m\frac{\text{arc } AB}{n} < \text{arc } CD \implies m(\text{arc } AB) < n(\text{arc } CD).$$

- If $m\alpha > n\beta$ then

$$m\frac{\alpha}{n} > \beta \implies m\gamma > \beta.$$

  Hence the circular sector cut out by β fits inside the circular sector cut out by $m\gamma$. It follows that

$$m(\text{arc } AB_1) > \text{arc } CD \implies m\frac{\text{arc } AB}{n} > \text{arc } CD \implies m(\text{arc } AB) > n(\text{arc } CD).$$

- Finally, if $m\alpha = n\beta$, then

$$m\frac{\alpha}{n} = \beta \implies m\gamma = \beta.$$

Hence the circular sector cut out by $m\gamma$ exactly overlaps the the circular sector cut out by β. It follows that

$$m(\text{arc } AB_1) = \text{arc } CD \implies m\frac{\text{arc } AB}{n} = \text{arc } CD \implies m(\text{arc } AB) = n(\text{arc } CD).$$

These three cases show that $m\alpha \overset{>}{\underset{<}{=}} n\beta$ exactly the same as $m(\text{arc } AB) \overset{>}{\underset{<}{=}} n(\text{arc } CD)$ for all $m, n \in \mathbb{Z}^+$. That is α is to β as arc AB is to arc CD.

(b) (5 pts) What prior result or results did you have to assume to be true in your proof of part (a)? How would you prove this result(s)?

I assumed that the following was already known: equal central angles intercept equal arcs in two circles of the same radii. This is easy enough to prove by arguing that if the central angles are equal and the radii are equal than the circular sectors intercepted by those central angles in the two circles must exactly overlap, which shows that the arcs are also equal.