Math 303 Exam 1 Solutions

Oct 12, 2009

1. (10 pts) Let A be a set with 7 elements and B a set with 5 elements. What can be said about the number of elements in the sets $A \cap B$ and $A \cup B$? Generalize to the case of any two finite sets A and B.

   $A \cap B$ contains those elements that are both in A and B. Since B has 5 elements, $0, 1, 2, \ldots, 5$ of these could be also in A. Therefore $0 \leq |A \cap B| \leq 5$.

   $A \cup B$ contains all the elements that are in A or B. Since A has 7 elements by itself, $A \cup B$ has to contain at least these. If all the elements of B are also elements of A, then $A \cup B$ contains exactly the 7 elements of A and no more. If on the other hand B has elements that are not in A, then these are also in $A \cup B$. There could be as many as 5 of these, so $A \cup B$ could have as many as 12 elements. Hence $7 \leq |A \cup B| \leq 12$.

   The above arguments readily generalize to show that $A \cap B$ could have as few as 0 elements, if A and B have nothing in common, or as many elements as the smaller of A and B. And of course any number inbetween. Hence

$$0 \leq |A \cap B| \leq \min(|A|, |B|).$$

   Similarly, $A \cup B$ must have at least as many elements as the larger of A and B and at most as many as $|A| + |B|$ if A and B have nothing in common. Therefore

$$\max(|A|, |B|) \leq |A \cup B| \leq |A| + |B|.$$

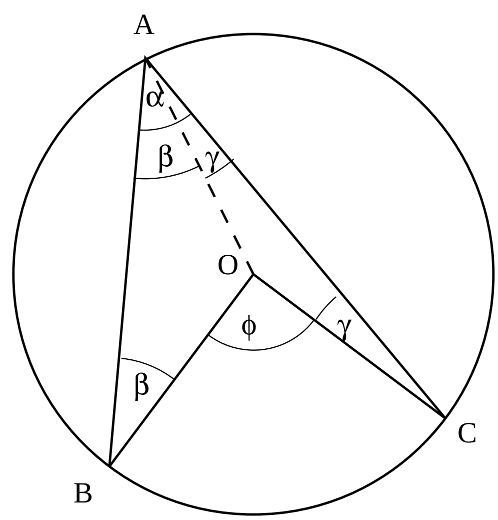
2. (10 pts) Assuming (1) a central angle of a circle is measured by its intercepted arc, (2) the sum of the angles of a triangle is equal to a straight angle, (3) the base angles of an isosceles triangle are equal, (4) a tangent to a circle is perpendicular to the radius drawn to the point of contact, prove that an angle inscribed in a circle is measured by one-half its intercepted arc.

   Refer to the picture on the right. We want to show that $\phi = 2\alpha$. First note that $\triangle ABO$ and $\triangle ACO$ are both isosceles because AO, BO, and CO are all radii of the circle. Hence by assumption (3), $\sphericalangle OAB = \sphericalangle ABO = \beta$ and $\sphericalangle OAC = \sphericalangle ACO = \gamma$. By assumption (2) applied to $\triangle ABO$,

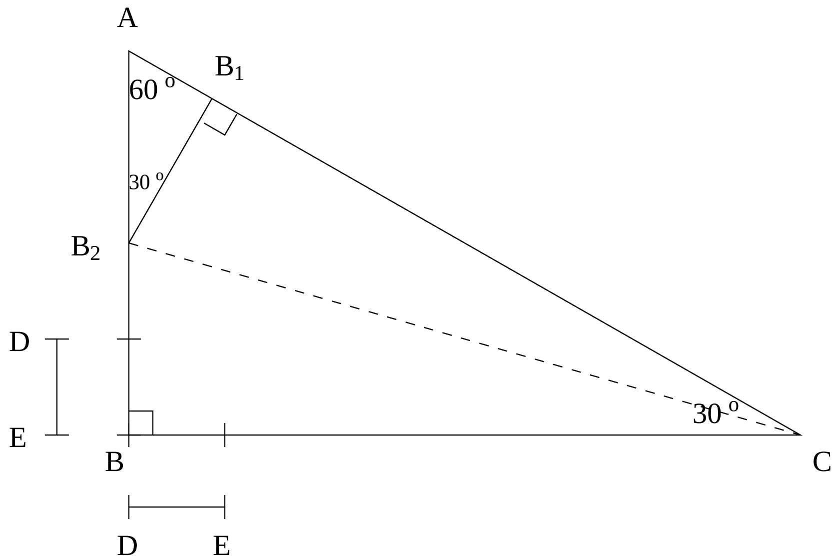
$$2\beta + \sphericalangle AOB = 180^\circ \implies \sphericalangle AOB = 180^\circ - 2\beta.$$

   Similarly, $\sphericalangle AOC = 180^\circ - 2\gamma$. Finally,

$$\begin{aligned} \phi + \sphericalangle AOB + \sphericalangle AOC &= 360^\circ \\ \phi + 180^\circ - 2\beta + 180^\circ - 2\gamma &= 360^\circ \\ \phi &= 2\beta + 2\gamma = 2\alpha \end{aligned}$$



3. (10 pts) Draw a $60^\circ - 30^\circ$ right triangle; mark off the longer leg, from the 30° angle vertex, on the hypotenuse; draw a perpendicular to the hypotenuse from the dividing point. Using this figure, formulate a geometric proof of the irrationality of $\sqrt{3}$.



Refer to the $60° - 30°$ right triangle above. Recall that if AB is of length 1, then AC is of length 2 and BC is of length $\sqrt{3}$.

This proof goes exactly like the geometric proof that $\sqrt{2}$ is irrational. Suppose that $\sqrt{3}$ is rational. Then AB and BC are commensurable. So there exists a line segment DE which fits into both AB and BC an integer number of times. Therefore DE fits into B_1C an integer number of times since $B_1C = BC$. Also DE fits into AC and integer number of times because $AC = 2AB$. AB_1 is the difference between AC and B_1C, so DE fits into AB_1 an integer number of times. Notice that $\triangle AB_1B_2$ is also a $60° - 30°$ right triangle. Therefore $AB_2 = 2AB_1$, which shows DE fits into AB_2 an integer number of times. BB_2 is the difference between AB and AB_2, hence DE fits into BB_2 an integer number of times.

Now note that $\triangle BB_2C$ and $\triangle B_1B_2C$ are congruent because they are both right angle triangles and have one leg and the hypotenuse in common, since $BC = B_1C$ is by construction. Therefore they have the other leg in common too by the Pythagorean Theorem. This shows $B_1B_2 = BB_2$ and then DE fits into B_1B_2 an integer number of times.

Now repeat this argument for the $60° - 30°$ right triangle AB_1B_2. If you continue this way, eventually you will get a triangle small enough that one of its sides is shorter than DE. But DE is supposed to fit into that side an integer number of times. This is a contradiction. Hence our assumption that $\sqrt{3}$ is rational must be false.

4. (a) (5 pts) Who was Pythagoras, and when and where did he live? (Approximate dates or even the correct century are sufficient.) What was his nationality?

   Pythagoras was an ancient Greek philosopher and mathematician who founded the Pythagorean brotherhood and is credited with the first proof of the Pythagorean Theorem. He lived about 572 BC - 495 BC first on the island of Samos, then in Crotona, present day Italy. (In case you wonder, he spent about half of his life in each place.)

   (b) (10 pts) State and prove the Pythagorean Theorem.

   In a right triangle the square of the hypotenuse equals the sum of the squares of the legs.
   See Chapter 4 of your textbook for nine different proofs. We gave a 10th one in class.

5. (a) (3 pts) Define what it means for two line segments AB and CD to be commensurable.

Two line segments $\overline{AB}$ and $\overline{CD}$ are commensurable if there exists a line segment $\overline{EF}$ which fits into each $\overline{AB}$ and $\overline{CD}$ an integer number of times.

(b) (12 pts) Prove that two line segments AB and CD are commensurable if and only if the ratio AB/CD is a rational number.

Let us do the "only if" part first. Suppose $\overline{AB}$ and $\overline{CD}$ are commensurable. Then there exists a line segment $\overline{EF}$ and two integers m, n such that $\overline{AB} = m\overline{EF}$ and $\overline{CD} = n\overline{EF}$. So

$$\frac{\overline{AB}}{\overline{CD}} = \frac{m\overline{EF}}{n\overline{EF}} = \frac{m}{n},$$

which is clearly a rational number.
Here is the "if" part. Suppose that $\overline{AB}/\overline{CD}$ is rational. That is there exist integers m, n such that $n \neq 0$ and $\overline{AB}/\overline{CD} = m/n$. Since the ratio of the lengths of two line segments must be a positive number, we may as well assume both m and n are positive. (If they are both negative, just switch their signs.) Now take $1/m$ part of $\overline{AB}$ and call it $\overline{EF}$. Now

$$\frac{\overline{AB}}{\overline{CD}} = \frac{m}{n} \implies \frac{\overline{AB}}{m} = \frac{\overline{CD}}{n} = \overline{EF}$$

shows that $\overline{EF}$ fits into $\overline{AB}$ m times and into $\overline{CD}$ n times.

6. **Extra credit problem.** The ancient Egyptians only knew fractions whose numerator is 1, e.g. $1/2, 1/3, 1/16$. These are called unit fractions. Whenever they had to write a rational number, they wrote it as an integer part plus a sum of unit fractions. E.g. $11/6 = 1+1/2+1/3$. It is clear that if you can write any rational number between 0 and 1 as a sum of unit fractions, then you can write any rational number as an integer plus a sum of unit fractions. In fact, if m/n is a rational number in the interval $[0, 1)$, it is easy enough to write it as a sum of unit fractions if we allow repetition of the same fraction. E.g.

$$\frac{17}{20} = \underbrace{\frac{1}{20} + \frac{1}{20} + \cdots + \frac{1}{20}}_{17 \text{ times}}.$$

But the Egyptians did not allow this. They would have written

$$\frac{17}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{10}$$

instead.

(a) (3 pts) Write 12/29 as a sum of distinct unit fractions.

One way is

$$\frac{12}{29} = \frac{1}{3} + \frac{1}{13} + \frac{1}{283} + \frac{1}{320073}.$$

(b) (12 pts) Prove that any rational number between 0 and 1 can be written as a sum of distinct unit fractions. (Hint: try a few more examples to find a pattern.)

To motivate my proof, let me first tell you how I came up with the decomposition of 12/29 as a sum of the unit fractions above. First, I found the largest unit fraction that is not larger than 12/29. This is 1/3, since $1/3 < 12/29 < 1/2$. Now, I looked at $12/29 - 1/3 = 7/87$ and found the largest unit fraction that is not larger than 7/87. This is 1/13, since $1/13 < 7/87 < 1/12$. Now I looked at $7/87 - 1/13 = 4/1131$ and found $1/283 < 4/1131 < 1/282$. Finally, $4/1131 - 1/283 = 1/320073$ which is a unit fraction. So I concluded

$$\frac{12}{29} - \frac{1}{3} - \frac{1}{13} - \frac{1}{283} = \frac{1}{320073} \implies \frac{12}{29} = \frac{1}{3} + \frac{1}{13} + \frac{1}{283} + \frac{1}{320073}.$$

The proof goes similarly. Start with a rational number m/n between 0 and 1. So we may assume m, n are both positive and $0 < m < n$. Divide n by m using integer division. You will get some quotient q_1 and some remainder r_1. Since r_1 is a remainder, we know $0 \leq r_1 < m$. Of course, we also know $n = q_1 m + r_1$. (Note for math majors: this is yet another use of the Division Algorithm.) If $r_1 = 0$, then $m/n = m/(q_1 m) = 1/q_1$ and we are done. Otherwise we will choose $1/(q_1 + 1)$ to be the first unit fraction in the decomposition. Note

$$\frac{m}{n} - \frac{1}{q_1+1} = \frac{(q_1+1)m - n}{(q_1+1)n} = \frac{(q_1+1)m - (q_1 m + r_1)}{(q_1+1)n} = \frac{m - r_1}{(q_1+1)n}.$$

Note that $m - r_1$ is smaller than m (but bigger than 0). Call this number m_1. Also, let $n_1 = (q_1 + 1)n$.

Now we will try writing m_1/n_1 as a sum of unit fractions. We will do this the same way. Divide n_1 by m_1 using integer division to find the quotient q_2 and r_2. So $n_1 = q_2 m_1 + r_2$ and $0 \leq r_2 < m_1$. If $r_2 = 0$, then

$$\frac{m_1}{n_1} = \frac{m_1}{q_2 m_1} = \frac{1}{q_2}$$

and hence

$$\frac{m}{n} = \frac{1}{q_1+1} + \frac{m_1}{n_1} = \frac{1}{q_1+1} + \frac{1}{q_2}$$

and we are done. If not, consider

$$\frac{m_1}{n_1} - \frac{1}{q_2+1} = \frac{(q_2+1)m_1 - n_1}{(q_2+1)n_1} = \frac{(q_2+1)m_1 - (q_2 m_1 + r_2)}{(q_2+1)n_1} = \frac{m_1 - r_2}{(q_2+1)n_1}.$$

Now let $m_2 = m_1 - r_2$ and note that $0 < m_2 < m_1$. Also, let $n_2 = (q_2 + 1)n_1$.

Keep repeating this procedure. Since the numerators of the fractions are getting smaller–remember $m > m_1 > m_2$–eventually, you will have to reach 1. And when you do, you are done writing the original rational number as a sum of unit fractions.

Does this argument make your head spin? Let's see how it works with 12/29. First, 29 divided by 12 is 2 and the remainder is 5. That remainder is not 0, so we are not done yet, but we will now look at

$$\frac{12}{29} - \frac{1}{2+1} = \frac{12}{29} - \frac{1}{3} = \frac{7}{87}.$$

87 divided by 7 is 12 and the remainder is 3. Still not 0, so we continue and look at

$$\frac{7}{87} - \frac{1}{12+1} = \frac{7}{87} - \frac{1}{13} = \frac{4}{1131}.$$

1131 divided by 4 is 282 and the remainder is 3. Still not 0, so we continue and look at

$$\frac{4}{1131} - \frac{1}{282+1} = \frac{4}{1131} - \frac{1}{283} = \frac{1}{320073}.$$

Actually, at this point we know we are done. But even if we failed to notice that for some reason, we would divide 320073 by 1 and we would get 320073 as the quotient and the remainder is 0. So this really is the end. Did you see how the numerator was decreasing? We started with 12, then we had 7, 4, 1.

Note to math majors: The above proof can be made shorter and more elegant by using strong induction on m. Give it a try.