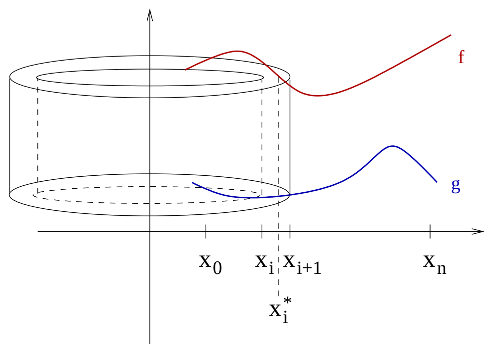
Math 1102 Final Exam Solutions

May 10, 2006

1. (20 pts)

(a) Let f and g be functions such that $g(x) \leq f(x)$ for $0 \leq a \leq x \leq b$. Let A be the area between the graphs of f and g from $x = a$ to $x = b$. Let V be the volume of revolution obtained by rotating A about the y-axis. Use a Riemann sum argument to explain how you would use integration to find the volume of V.



We split $[a, b]$ into n intervals of length Δx by letting $x_0 = a$, $x_n = b$ and $x_1, \ldots, x_{n-1}$ be evenly spaced points inbetween. When rotated about the y-axis, the two graphs on such an interval almost make a cylindrical shell. Let $x_i^* \in [x_i, x_{i+1}]$. The inner radius of the i-th such shell is x_i and its height is approximately $f(x_i^*) - g(x_i^*)$. So its volume is approximately

$$2\pi x_i(f(x_i^*) - g(x_i^*))$$

We need to add these and let $\Delta x \to 0$, so the approximation becomes perfectly accurate:

$$V = \lim_{\Delta x \to 0} \sum_{i=0}^{n} 2\pi x_i(f(x_i^*) - g(x_i^*)) = \int_a^b 2\pi x(f(x) - g(x))\, dx$$

(b) Use vertical cylindrical shells to find the volume of a sphere of radius r centered at the origin.

To get a sphere of radius r, we rotate a semicircle about the y-axis. We can find the volume by using the result from the previous part with $f(x) = \sqrt{r^2 - x^2}$ and $g(x) = -\sqrt{r^2 - x^2}$ on $x \in [0, r]$.

$$V = \int_0^r 2\pi x(\sqrt{r^2 - x^2} + \sqrt{r^2 - x^2})\, dx = 4\pi \int_0^r x\sqrt{r^2 - x^2}\, dx$$
$$= -2\pi \int_0^r -2x\sqrt{r^2 - x^2}\, dx = -2\pi \left[\frac{2}{3}(r^2 - x^2)^{3/2}\right]_0^r = -\frac{4}{3}\pi(0 - r^3) = \frac{4}{3}\pi r^3$$

(c) Use horizontal slices to find the volume of a sphere of radius r centered at the origin.

The horizontal slices will be thin disks of radius $\sqrt{r^2 - y^2}$. So

$$V = \int_{-r}^r \pi \sqrt{r^2 - y^2}^2\, dy = \pi \int_{-r}^r r^2 - y^2\, dy$$
$$= \pi \left[r^2 y - \frac{y^3}{3}\right]_{-r}^r = \pi \left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3$$

2. (15 pts) Consider

$$\int_{-2}^{-1} \frac{1}{x\sqrt{x^2 - 1}}\, dx$$

(a) Why is this an improper integral?

Because at $x = -1$, the denominator is 0, so the integrand is discontinuous.

(b) Evaluate the integral if possible.

First note that

$$\int_{-2}^{-1} \frac{1}{x\sqrt{x^2-1}}\, dx = \lim_{t\to -1^-} \int_{-2}^{t} \frac{1}{x\sqrt{x^2-1}}\, dx$$

We will substitute $x = \sec(u)$. This makes $dx = \sec(u)\tan(u)du$. We will leave the limits in x.

$$\begin{aligned}\lim_{t\to -1^-} \int_{-2}^{t} \frac{1}{x\sqrt{x^2-1}}\, dx &= \lim_{t\to -1^-} \int_{x=-2}^{x=t} \frac{1}{\sec(u)\sqrt{\sec^2(u)-1}} \sec(u)\tan(u)\, du \\ &= \lim_{t\to -1^-} \int_{x=-2}^{x=t} \frac{\tan(u)}{\sqrt{\tan^2(u)}}\, du\end{aligned}$$

As x goes from -2 to -1, u goes from $\operatorname{arcsec}(-2) = \arccos(-1/2) = 2\pi/3$ to $\operatorname{arcsec}(-1) = \arccos(-1) = \pi$. Note that $\tan(u) \le 0$ on this interval, so $\sqrt{\tan^2(u)} = -\tan(u)$.

$$\begin{aligned}\lim_{t\to -1^-} \int_{-2}^{t} \frac{1}{x\sqrt{x^2-1}}\, dx &= \lim_{t\to \pi^-} \int_{x=-2}^{x=t} -1\, du \\ &= \lim_{t\to -1^-} [-u]_{x=-2}^{x=t} = \lim_{t\to -1^-} [-\operatorname{arcsec}(x)]_{-2}^{t} \\ &= \lim_{t\to -1^-} (\operatorname{arcsec}(-2) - \operatorname{arcsec}(t)) = \operatorname{arcsec}(-2) - \operatorname{arcsec}(-1) \\ &= \frac{2}{3}\pi - \pi = -\frac{\pi}{3}\end{aligned}$$

(c) Is your result consistent with the sign of the integrand?

Yes, it is because the integrand is clearly negative for $x \in [-2, -1)$. So the integral should also be negative.

3. (10 pts)

(a) Find the area under the parametric curve

$$\begin{aligned}x(t) &= \sin(t) \\ y(t) &= \cos^2(t)\end{aligned}$$

from $t = 0$ to $t = \pi/2$.

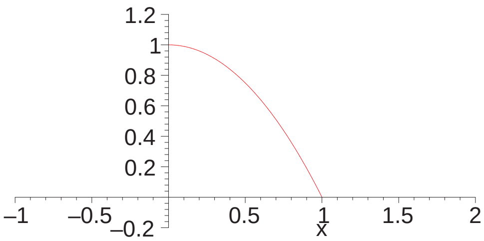
$$\begin{aligned}\int_0^{\pi/2} y\frac{dx}{dt}\, dt &= \int_0^{\pi/2} \cos^2(t)\cos(t)\, dt = \int_0^{\pi/2} \cos^3(t)\, dt \\ &= \int_0^{\pi/2} \cos(t)(1-\sin^2(t))\, dt = \int_0^{\pi/2} \cos(t) - \cos(t)\sin^2(t)\, dt \\ &= \left[\sin(t) - \frac{\sin^3(t)}{3}\right]_0^{\pi/2} = 1 - \frac{1}{3} = \frac{2}{3}\end{aligned}$$

(b) Find an equation of this curve in Cartesian coordinates and sketch the curve.

Notice that $x^2 = \sin^2(t)$, so

$x^2 + y = \cos^2(t) + \sin^2(t) = 1 \implies y = 1 - x^2$

As t goes from 0 to $\pi/2$, $x = \sin(t)$ goes from 0 to 1. The graph is on the right.

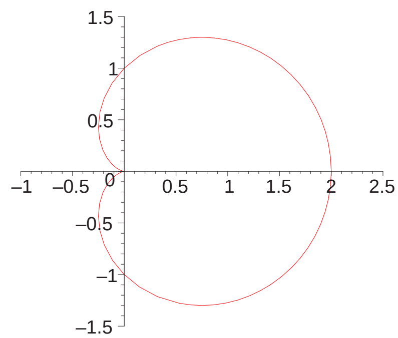


4. (10 pts) Consider the polar curve

$$r = 1 + \cos(\theta)$$
$$0 \le \theta \le 2\pi$$

(a) Sketch the curve.

This is another cardioid.



(b) Find its arc length using integration.

$$\begin{aligned}
\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^{2\pi} \sqrt{(1+\cos(\theta))^2 + (-\sin(\theta))^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{1 + 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\cos(\theta)}\, d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1 + 1\cos(\theta)}\, d\theta \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 2\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta \\
&= 2\left(\int_0^{\pi} \cos\left(\frac{\theta}{2}\right)\, d\theta + \int_{\pi}^{2\pi} -\cos\left(\frac{\theta}{2}\right)\, d\theta\right) \\
&= 4\left(\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} - \left[\sin\left(\frac{\theta}{2}\right)\right]_{\pi}^{2\pi}\right) = 4(1+1) = 8
\end{aligned}$$

5. (15 pts)

(a) Solve the first order linear differential equation

$$\frac{dy}{dx} + y = \sin(x)$$
$$y(0) = 1$$

The integrating factor is $I(x) = e^{\int 1\,dx} = e^x$. So

$$\begin{aligned}
e^x \frac{dy}{dx} + e^x y &= e^x \sin(x) \\
\frac{d}{dx}(e^x y) &= e^x \sin(x) \\
e^x y &= \int e^x \sin(x)\,dx
\end{aligned}$$

To integrate $e^x \sin(x)$, we do integration by parts twice

$$\begin{aligned}
\int e^x \sin(x)\,dx &= e^x \sin(x) - \int e^x \cos(x)\,dx \\
&= e^x \sin(x) - \left(e^x \cos(x) - \int e^x(-\sin(x))\,dx\right) \\
&= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\,dx
\end{aligned}$$

We solve for $\int e^x \sin(x)\,dx$ by adding it to both sides and dividing by 2:

$$\begin{aligned}
2\int e^x \sin(x)\,dx &= e^x \sin(x) - e^x \cos(x) + C \\
\int e^x \sin(x)\,dx &= \frac{e^x}{2}(\sin(x) - \cos(x)) + \frac{C}{2}
\end{aligned}$$

We can of course replace $C/2$ with C as usual. So

$$\begin{aligned}
e^x y &= \frac{e^x}{2}(\sin(x) - \cos(x)) + C \\
y &= \frac{\sin(x) - \cos(x)}{2} + Ce^{-x}
\end{aligned}$$

We can solve for C using the initial condition:

$$1 = y(0) = \frac{\sin(0) - \cos(0)}{2} + Ce^0 = -\frac{1}{2} + C \implies C = \frac{3}{2}$$

So the particular solution of the differential equation is

$$y = \frac{\sin(x) - \cos(x)}{2} + \frac{3}{2}e^{-x}$$

(b) Check that your answer satisfies both the differential equation and the initial value.

$$\begin{aligned}
\frac{dy}{dx} + y &= \frac{\cos(x) + \sin(x)}{2} - \frac{3}{2}e^{-x} + \frac{\sin(x) - \cos(x)}{2} + \frac{3}{2}e^{-x} = \sin(x) \qquad \checkmark \\
y(0) &= \frac{\sin(0) - \cos(0)}{2} + \frac{3}{2}e^0 = -\frac{1}{2} + \frac{3}{2} = 1 \qquad \checkmark
\end{aligned}$$

6. (20 pts) A team of biologists wants to model a population of howler monkeys on an island. They observe that the rate of growth of the population is approximately proportional to the number of monkeys on the island, unless there are too many or two few monkeys. If there are too many monkeys, they spend too much time arguing about who gets to eat the bananas; if there are too few monkeys, there isn't enough monkey love going on to grow the population. In fact, if the population drops below a certain threshold, its rate of growth becomes negative and the number of monkeys declines. Such behavior is modeled by the differential equation
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)\left(1 - \frac{m}{P}\right)$$
where L and m are constants.

(a) Find the equilibrium solutions of this differential equation.

We have an equilibrium when $dP/dt = 0$. Since the right-hand side is already factored, we can see this will happen when
$$P = 0 \text{ or } 1 - \frac{P}{L} = 0 \text{ or } 1 - \frac{m}{P} = 0$$
that is when $P = 0$ or $P = L$ or $P = m$.

(b) What is the threshold below which the population declines?

Notice
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)\left(1 - \frac{m}{P}\right) = \frac{k}{L}(L - P)(P - m)$$
We know $dP/dt < 0$ if P is large or if P is small. If P is large enough, $L - P < 0$ and $P - m > 0$. Assuming $k, L > 0$, as in the logistic model, this would make $dP/dt < 0$. On the other hand, if P is small enough, $L - P > 0$ and $P - m < 0$, making $dP/dt < 0$ $dP/dt < 0$. For this to happen, P must be less than m, so m is such a threshold.

(c) Find the general solution of this differential equation. Make sure to specify what kind of values any arbitrary constant can take on.

We will do this exactly the same way as we solved the logistic equation. First, note that the equation is separable, so we can use the usual separation trick.
$$\frac{dP}{dt} = \frac{k}{L}(L - P)(P - m)$$
$$\int \frac{dP}{(L - P)(P - m)} = \int \frac{k}{L}\,dt = \frac{k}{L}t + C$$
We can find the indefinite integral on the left by using partial fractions:
$$\frac{1}{(L - P)(P - m)} = \frac{A}{L - P} + \frac{B}{P - m}$$
$$1 = A(P - m) + B(L - P)$$
If $P = L$,
$$1 = A(L - m) \implies A = \frac{1}{L - m}$$

If $P = m$,

$$1 = B(L - m) \implies B = \frac{1}{L - m}$$

So

$$\begin{aligned}
\int \frac{dP}{(L-P)(P-m)} &= \frac{1}{L-m} \int \frac{1}{L-P} + \frac{1}{P-m}\, dP \\
&= \frac{1}{L-m}(-\ln|L-P| + \ln|P-m|) \\
&= \frac{1}{L-m} \ln\left|\frac{P-m}{L-P}\right|
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{L-m} \ln\left|\frac{P-m}{L-P}\right| &= \frac{k}{L}t + C && C \in \mathbb{R} \\
\ln\left|\frac{P-m}{L-P}\right| &= \frac{k(L-m)}{L}t + \underbrace{C(L-m)}_{D} && D \in \mathbb{R} \\
\left|\frac{P-m}{L-P}\right| &= e^{\frac{k(L-m)}{L}t} \underbrace{e^{D}}_{E} && E > 0 \\
\frac{P-m}{L-P} &= \underbrace{\pm E}_{F} e^{\frac{k(L-m)}{L}t} && F \neq 0 \\
P - m &= Fe^{\frac{k(L-m)}{L}t}(L - P) \\
P\left(Fe^{\frac{k(L-m)}{L}t} + 1\right) &= Fe^{\frac{k(L-m)}{L}t}L + m \\
P &= \frac{FLe^{\frac{k(L-m)}{L}t} + m}{Fe^{\frac{k(L-m)}{L}t} + 1}
\end{aligned}$$

Notice that if $F = 0$, then $P = m$, which we already said was a perfectly fine equilibrium solution, so the arbitrary constant F could be any real number.

(d) Check your answer by substituting it back in the differential equation.

First, the left-hand side is

$$\begin{aligned}
\frac{k}{L}(L-P)(P-m) &= \frac{k}{L}\left(L - \frac{FLe^{\frac{k(L-m)}{L}t} + m}{Fe^{\frac{k(L-m)}{L}t} + 1}\right)\left(\frac{FLe^{\frac{k(L-m)}{L}t} + m}{Fe^{\frac{k(L-m)}{L}t} + 1} - m\right) \\
&= \frac{k}{L} \frac{L\left(Fe^{\frac{k(L-m)}{L}t} + 1\right) - \left(FLe^{\frac{k(L-m)}{L}t} + m\right)}{Fe^{\frac{k(L-m)}{L}t} + 1} \frac{FLe^{\frac{k(L-m)}{L}t} + m - m\left(Fe^{\frac{k(L-m)}{L}t} + 1\right)}{Fe^{\frac{k(L-m)}{L}t} + 1} \\
&= \frac{k}{L} \frac{FLe^{\frac{k(L-m)}{L}t} + L - FLe^{\frac{k(L-m)}{L}t} - m}{Fe^{\frac{k(L-m)}{L}t} + 1} \frac{FLe^{\frac{k(L-m)}{L}t} + m - Fme^{\frac{k(L-m)}{L}t} - m}{Fe^{\frac{k(L-m)}{L}t} + 1} \\
&= \frac{k}{L} \frac{L-m}{Fe^{\frac{k(L-m)}{L}t} + 1} \frac{Fe^{\frac{k(L-m)}{L}t}(L-m)}{Fe^{\frac{k(L-m)}{L}t} + 1} \\
&= F\frac{k}{L}(L-m)^2 \frac{e^{\frac{k(L-m)}{L}t}}{\left(Fe^{\frac{k(L-m)}{L}t} + 1\right)^2}
\end{aligned}$$

The right-hand side is

$$\frac{dP}{dt} = \frac{\left(Fk(L-m)e^{\frac{k(L-m)}{L}t}\right)\left(Fe^{\frac{k(L-m)}{L}t}+1\right) - \left(FLe^{\frac{k(L-m)}{L}t}L+m\right)\left(F\frac{k(L-m)}{L}e^{\frac{k(L-m)}{L}t}\right)}{\left(Fe^{\frac{k(L-m)}{L}t}+1\right)^2}$$

$$= \frac{F^2k(L-m)e^{2\frac{k(L-m)}{L}t} + Fk(L-m)e^{\frac{k(L-m)}{L}t} - F^2k(L-m)e^{2\frac{k(L-m)}{L}t} - Fm\frac{k(L-m)}{L}e^{\frac{k(L-m)}{L}t}}{\left(Fe^{\frac{k(L-m)}{L}t}+1\right)^2}$$

$$= \frac{Fk(L-m)e^{\frac{k(L-m)}{L}t}\left(1-\frac{m}{L}\right)}{\left(Fe^{\frac{k(L-m)}{L}t}+1\right)^2} = Fk(L-m)\frac{L-m}{L}\frac{e^{\frac{k(L-m)}{L}t}}{\left(Fe^{\frac{k(L-m)}{L}t}+1\right)^2}$$

which we see is the same as the left-hand side.

7. (10 pts)

(a) Prove that if the series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Let

$$s_n = \sum_{k=1}^{n} a_k$$

If the series is convergent, then there is some number $s = \lim_{n\to\infty} s_n$. Notice that

$$s_n - s_{n-1} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = a_n$$

Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

(b) Is the converse true? If yes, prove it, if not, give a counterexample and justify it.

The converse is false. Look at the harmonic series:

$$\sum_{n=1}^{\infty} = \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

which of course gets arbitrarily large, so the series diverges. But

$$\lim_{n\to\infty} \frac{1}{n} = 0$$