Math 1102 Exam 3 Solutions

Apr 7, 2006

1. (15 pts)

(a) State the definition of $\ln(x)$ in terms of an appropriate integral. Be sure to specify for what values of x your are defining $\ln(x)$.

For $x > 0$,

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

(b) Use the above definition to show that for all $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$.

$$\ln(x^y) = y\ln(x)$$

By the FTC,

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

Hence

$$\frac{d}{dx}\ln(x^y) = \frac{1}{x^y}yx^{y-1} = \frac{y}{x} = \frac{d}{dx}y\ln(x)$$

This is true for all $x \in (0,\infty)$. Two functions that are differentiable on an interval and have the same derivative, differ by a constant on that interval. Hence, for some C

$$\ln(x^y) = y\ln(x) + C$$

To find C, note that

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$$

so

$$0 = \ln(1^y) = y\ln(1) + C = 0 + C \implies C = 0$$

which shows

$$\ln(x^y) = y\ln(x)$$

for all $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$.

(c) Using the above definition of ln, prove that

$$\lim_{x\to\infty}\ln(x) = \infty$$

First, note that ln is an increasing function on $(0,\infty)$ because its derivative is $1/x$, which is positive.
Second, note that

$$\ln(2^y) = y\ln(2) = y\int_1^2 \frac{1}{t}\,dt$$

Since $1/t$ is positive on $t \in [1,2]$, the integral is positive. So $\ln(2^y)$ can be made arbitrarily big by choosing y to be sufficiently big. Now, for any $x > 2^y$, $\ln(x) \geq \ln(2^y)$. So we can be made arbitrarily big by for any x sufficiently large (larger than 2^y). Hence

$$\lim_{x\to\infty}\ln(x) = \infty$$

2. (10 pts) Consider the graph of $f(x) = \cosh(x)$ for $-1 \le x \le 1$. Find the area of the surface you get by rotating this graph about the x-axis. (Remember $\cosh(x) = (e^x + e^{-x})/2$.)

$$\begin{aligned}
\int_{-1}^{1} 2\pi \cosh(x)\sqrt{1+\sinh^2(x)}\,dx &= 2\pi \int_{-1}^{1} \cosh(x)\sqrt{\cosh^2(x)}\,dx \\
&= 2\pi \int_{-1}^{1} \cosh(x)\cosh(x)\,dx \\
&= 2\pi \int_{-1}^{1} \cosh^2(x)\,dx \\
&= 2\pi \int_{-1}^{1} \left(\frac{e^x + e^{-x}}{2}\right)\,dx \\
&= 2\pi \int_{-1}^{1} \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4}\,dx \\
&= \frac{\pi}{2}\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_{-1}^{1} = \pi\frac{e^2 - e^{-2}}{2} + 2\pi
\end{aligned}$$

where we used $\sqrt{\cosh^2(x)} = \cosh(x)$ because $\cosh(x) \ge 1$ for all x.

3. (10 pts each) Do the following improper integrals converge? Evaluate those that do.

(a) $\displaystyle\int_0^{\pi} \tan(x)\,dx$

Notice that $\tan(x)$ has a vertical asymptote at $x = \pi/2$. So

$$\begin{aligned}
\int_0^{\pi} \tan(x)\,dx &= \int_0^{\pi/2} \tan(x)\,dx + \int_{\pi/2}^{\pi} \tan(x)\,dx \\
&= \lim_{t\to\frac{\pi}{2}-} \int_0^{t} \tan(x)\,dx + \lim_{t\to\frac{\pi}{2}+} \int_t^{\pi} \tan(x)\,dx \\
&= \lim_{t\to\frac{\pi}{2}-} \ln|\sec(x)|\Big|_0^t + \lim_{t\to\frac{\pi}{2}+} \ln|\sec(x)|\Big|_t^{\pi}
\end{aligned}$$

But

$$\lim_{t\to\frac{\pi}{2}-} \ln|\sec(t)| = \infty$$

so this integral is divergent.

(b) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\,dx$

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\,dx &= \int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx + \int_0^{1} \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{-x}}\,dx + \int_0^{1} \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{t\to 0-} \int_{-1}^{t} \frac{1}{\sqrt{-x}}\,dx + \lim_{t\to 0+} \int_t^{1} \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{t\to 0-} -2\sqrt{-x}\Big|_{-1}^{t} + \lim_{t\to 0+} 2\sqrt{x}\Big|_t^{1} = 0 - (-2) + 2 - 0 = 4
\end{aligned}$$

4. (15 pts) Lt. Columbo is investigating a bank robbery, which took place at 4PM. He visits the suspect's home at 8PM. The suspect's friends assure him that they (incl. the suspect) were watching the Smurfs on video all day long, and they didn't even leave the house since breakfast. On his way out, Columbo notices that the suspect's car feels warm. He measures the temperature of the coolant and it's 25°C. An hour later, he sneaks back to take another measurement. It is now 20°C. The thermostat in the garage is set to 15°C. He knows that the normal coolant temperature in a running engine is 95°C.

(a) Assuming that Newton's Law of Cooling applies, find a differential equation to describe the temperature of the coolant in the car.

The differential equation is the usual: the rate of change of the temperature of the coolant is proportional to the temperature difference between it and its environment. We will choose $t = 0$ to be the time Columbo first checks the temperature.

$$\begin{aligned}\frac{dT}{dt} &= k(T - 15^\circ\text{ C})\\ T(0) &= 25^\circ\text{ C}\\ T(1) &= 20^\circ\text{ C}\end{aligned}$$

(b) Solve the equation, and use the given data to figure out the constants in your solution.

$$\begin{aligned}\frac{dT}{dt} &= k(T - 15)\\ \int \frac{dT}{T-15} &= \int k\,dt\\ \ln|T-15| &= kt + C\\ |T-15| &= e^{kt+C} = Ae^{kt} && \text{where } A = e^C > 0\\ T - 15 &= \pm Ae^{kt}\\ T &= 15 + Be^{kt} && \text{where } B = \pm A \neq 0\end{aligned}$$

We can determine B (actually $B = 0$ would be ok) and k using the initial conditions:

$$T(0) = 25 \implies 25 = 15 + B \implies B = 10^\circ\text{ C}$$

$$T(1) = 20 \implies 20 = 15 + 10e^k \implies k = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

So

$$T(t) = 15 + 10e^{-t\ln(2)} = 15 + 10\left(\frac{1}{2}\right)^t$$

(c) Should Columbo believe the suspect and his friends?

Let's see when the car was last used.

$$95 = 15 + 10\left(\frac{1}{2}\right)^t \implies 8 = \left(\frac{1}{2}\right)^t \implies t = -3$$

So at 1 PM. This makes the suspect's story suspect.

5. (15 pts) **Extra credit problem.** Don't attempt this problem until you are done with everything else.

Let $f(x) = 1/x^2$. Consider the object obtained by rotating the graph of f for $1 \leq x$ about the x-axis.

(a) Show that this seemingly infinite object has finite surface area.

$$\int_1^\infty 2\pi \frac{1}{x^2}\sqrt{1+\left(\frac{-2}{x^3}\right)^2}\,dx = 2\pi \int_1^\infty \frac{1}{x^2}\sqrt{1+\frac{4}{x^6}}\,dx$$

This integral is hard to evaluate, but we can use the Comparison Theorem. If $x \geq 1$,

$$\frac{4}{x^6} \leq 4 \implies \sqrt{1+\frac{4}{x^6}} \leq \sqrt{5} \implies \frac{1}{x^2}\sqrt{1+\frac{4}{x^6}} \leq \frac{1\sqrt{5}}{x^2}$$

We know

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = 1$$

Hence

$$2\pi \int_1^\infty \frac{1}{x^2}\sqrt{1+\frac{4}{x^6}}\,dx \leq 2\pi \int_1^\infty \frac{\sqrt{5}}{x^2} = 2\pi\sqrt{5}$$

So the surface area is finite.

(b) What can you say about the arc length of the graph of f for $1 \leq x$?

The arc length is

$$\int_1^\infty \sqrt{1+\frac{4}{x^6}}\,dx$$

which again may be hard to compute. But obviously,

$$1 \leq \sqrt{1+\frac{4}{x^6}}$$

and

$$\int_1^\infty 1\,dx = \infty$$

so by the Comparison Theorem, the arc length is ∞.

(c) Do these results make sense to you? Why?

The results may strike you as curious. But the reason why we get an object of finite surface area by revolving an infinite curve about the x-axis is that the curve quickly gets very close to the x-axis. It's infinitely long, but very very thin.