Math 1102 Exam 2 Solutions

Mar 1, 2006

1. (15 pts)

(a) State the definition of $\ln(x)$ in terms of an appropriate integral. Be sure to specify for what values of x your are defining $\ln(x)$.

For $x > 0$,

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

(b) Use the above definition to show that for all $x, y > 0$,

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y).$$

By the FTC,

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

Notice that

$$\frac{d}{dx}\ln\left(\frac{x}{y}\right) = \frac{y}{x}\frac{1}{y} = \frac{1}{x} = \frac{d}{dx}\ln(x)$$

This is true for all $x > 0$, that is on the interval $(0, \infty)$. We know that two functions whose derivatives are equal on an interval can only differ by a constant on that interval. Hence

$$\ln\left(\frac{x}{y}\right) = \ln(x) + C$$

We can determine C by setting $x = y$. Notice

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$$

where the integral is 0 because the upper and lower bounds are the same.

$$0 = \ln(1) = \ln\left(\frac{y}{y}\right) = \ln(y) + C \implies C = -\ln(y)$$

Hence

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y).$$

2. (20 pts)

(a) Use a trig substitution and then integration by parts to find

$$\int \sqrt{x^2+1}\,dx$$

Let's substitute $x = \tan(u)$. Then $dx = \sec^2(u)du$. In order to have an inverse, we will restrict $u \in (-\pi/2, \pi/2)$. This is no restriction at all on x. Then $u = \arctan(x)$.

$$\int \sqrt{x^2+1}\,dx = \int \sqrt{\tan^2(u)+1}\,\sec^2(u)\,du = \int \sqrt{\sec^2(u)}\,\sec^2(u)\,du$$

Since $u \in (-\pi/2, \pi/2)$, $\sec(u) = 1/\cos(u) > 0$, so $\sqrt{\sec^2(u)} = \sec(u)$.

$$\int \sqrt{\sec^2(u)}\, \sec^2(u)\, du = \int \sec^3(u)\, du = \int \sec(u)\sec^2(u)\, du$$

We will now use integration by parts with $f = \sec(u)$ and $g' = \sec^2(u)$. Then $f' = \sec(u)\tan(u)$ and $g = \tan(u)$ and

$$\begin{aligned}
\int \sec(u)\sec^2(u)\, du &= \sec(u)\tan(u) - \int \sec(u)\tan^2(u)\, du \\
&= \sec(u)\tan(u) - \int \sec(u)(\sec^2(u) - 1)\, du \\
&= \sec(u)\tan(u) - \int \sec^3(u)\, du + \int \sec(u)\, du \\
&= \sec(u)\tan(u) - \int \sec^3(u)\, du + \ln|\sec(u) + \tan(u)|
\end{aligned}$$

Now add $\int \sec^3(u)\, du$ to both sides and solve for it.

$$\begin{aligned}
2\int \sec^3(u)\, du &= \sec(u)\tan(u) + \ln|\sec(u) + \tan(u)| \\
\int \sec^3(u)\, du &= \frac{\sec(u)\tan(u) + \ln|\sec(u) + \tan(u)|}{2} + C
\end{aligned}$$

where the constant C was added because we have an indefinite integral. We could now use a right-angle triangle to find $\sec(u)$, so just the fact that $\sec^2(u) = \tan^2(u) + 1 = x^2 + 1$, and $\sec(u) > 0$, so $\sec(u) = \sqrt{x^2 + 1}$. Hence

$$\int \sqrt{x^2 + 1}\, dx = \frac{x\sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}|}{2} + C$$

Notice that we can easily check this answer.

$$\begin{aligned}
\frac{d}{dx}\frac{x\sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}|}{2} &= \frac{1}{2}\left(\sqrt{x^2 + 1} + x\frac{1}{2\sqrt{x^2 + 1}}(2x) + \frac{1 + \frac{1}{2\sqrt{x^2+1}}(2x)}{x + \sqrt{x^2 + 1}}\right) \\
&= \frac{1}{2}\left(\frac{x^2 + 1 + x^2 + \frac{\sqrt{x^2+1}+x}{x+\sqrt{x^2+1}}}{\sqrt{x^2 + 1}}\right) \\
&= \frac{1}{2}\frac{2 + 2x^2}{\sqrt{x^2 + 1}} = \sqrt{x^2 + 1}
\end{aligned}$$

(b) Evaluate the following integral

$$\int \frac{x^2 + 3}{(x + 1)^2(x + 3)}\, dx$$

We will use partial fractions.

$$\begin{aligned}
\frac{x^2 + 3}{(x + 1)^2(x + 3)} &= \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x + 3} \\
x^2 + 3 &= A(x + 1)(x + 3) + B(x + 3) + C(x + 1)^2
\end{aligned}$$

Substituting $x = -1$,

$$(-1)^2 + 3 = 2B \implies B = 2$$

Substituting $x = -3$,

$$(-3)^2 + 3 = 4C \implies C = 3$$

Finally substituting any other value, for example $x = 0$,

$$0^2 + 3 = 3A + 3B + C = 3A + 6 + 3 \implies A = -2$$

Hence

$$\begin{aligned}\int \frac{x^2+3}{(x+1)^2(x+3)}\,dx &= \int \frac{-2}{x+1} + \frac{2}{(x+1)^2} + \frac{3}{x+3}\,dx \\ &= -2\ln|x+1| + 2\frac{(x+1)^{-1}}{-1} + 3\ln|x+3| + C \\ &= -2\ln|x+1| - 2\frac{1}{x+1} + 3\ln|x+3| + C\end{aligned}$$

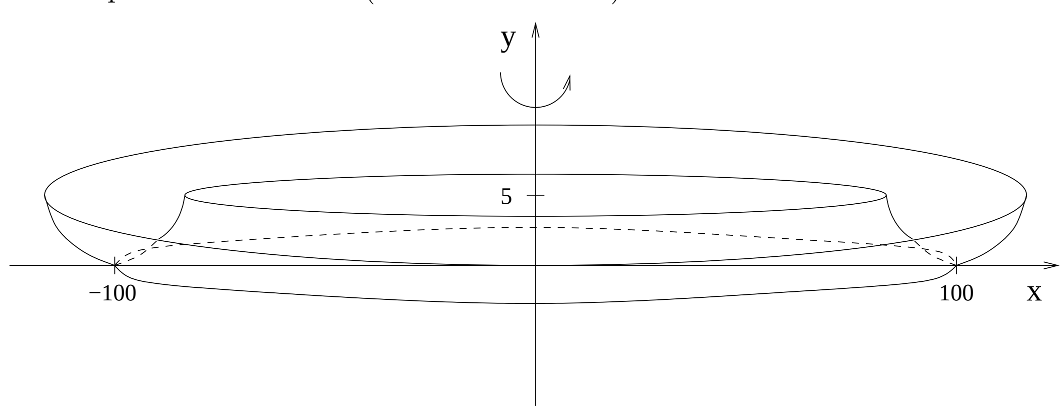
Again, this is easy to check

$$\begin{aligned}\frac{d}{dx} - 2\ln|x+1| - 2\frac{1}{x+1} + 3\ln|x+3| &= \frac{-2}{x+1} + \frac{2}{(x+1)^2} + \frac{3}{x+3} \\ &= \frac{-2(x+1)(x+3) + 2(x+3) + 3(x+1)^2}{(x+1)^2(x+3)} \\ &= \frac{-2x^2 - 8x - 6 + 2x + 6 + 3x^2 + 6x + 3}{(x+1)^2(x+3)} \\ &= \frac{x^2+3}{(x+1)^2(x+3)}\end{aligned}$$

3. (15 pts) Sauron, the Dark Lord of Mordor lives in a castle surrounded by a moat filled with fetid swampwater infested with crocodiles. One day his safety inspector points out to him that according to the latest statistics, orcs who accidentally fall into the moat tend to experience rapid decline in their health. This makes for an unwholesome work environment for his loyal underlings. Therefore the moat has to be emptied of the hazardous liquid.

   The shape of the moat is determined by the parabola $(x-100)^2/5$ rotated about the vertical axis. It is 10 m wide and 5 m deep and is filled to the rim. How much work does it take to pump out the contents of the moat? According to Sauron's book "Physics for the practicing wizard" the density of fetid swampwater infested with crocodiles is $1500\,\mathrm{kg/m^3}$ and for all practical purposes $g = 10, \mathrm{m/s^2}$.

   Here is a picture of the moat (not drawn to scale).



We will divide the water in the moat into horizontal slices. Such a slize has the shape of a washer. The inner and out radii are determined by

$$\begin{aligned} y &= (x-100)^2/5 \\ 5y &= (x-100)^2 \\ \pm\sqrt{5y} &= x - 100 \\ x &= 100 \pm \sqrt{5y} \end{aligned}$$

So the volume of a slice is

$$V = (\pi(100+\sqrt{5y})^2 - \pi(100-\sqrt{5y})^2)\Delta y = 400\pi\sqrt{5y}\Delta y$$

We get the mass of the slice by multiplying by the density and the weight by multiplying by g. The slize needs to be moved up to the rim, so by $5-y$. Hence the work is

$$\begin{aligned} \int_0^5 1500 \cdot 10 \cdot 400\pi\sqrt{5y}(5-y)\,dy &= 6000000\sqrt{5}\pi \int_0^5 5y^{1/2} - y^{3/2}\,dy \\ &= 6000000\sqrt{5}\pi\left[5\frac{2y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^5 \\ &= 6000000\sqrt{5}\pi\left(\frac{10}{3}5^{3/2} - \frac{2}{5}5^{5/2}\right) \\ &= 6000000\sqrt{5}\pi\left(\frac{50}{3} - 10\right)\sqrt{5} \\ &= 30000000\pi\frac{20}{3} = 200000000\pi \end{aligned}$$

Multiplying the units, we get

$$\frac{\mathrm{kg}}{\mathrm{m}^3}\frac{\mathrm{m}}{\mathrm{s}^2}\,\mathrm{m}^2\,\mathrm{m}\,\mathrm{m} = \frac{\mathrm{kgm}^2}{\mathrm{s}^2} = \mathrm{J}$$

So the work it takes to pump the water out is 200000000π Joules, or 200π MJ ≈ 628 MJ.