Math 1102 Exam 1 Solutions

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1. (15 pts)
   (a) State the Fundamental Theorem of Calculus.

   Let f be a continous function on the interval $[a, b]$ and F an antiderivative of f. Then
   1. $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ for $x \in [a, b]$.
   2. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

   (b) Decide if the following statement is true or false. If true, prove it, if false, give a counterexample. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x) \geq 0$ for all $x \in \mathbb{R}$, then

$$\int_a^b f(x)\,dx \geq 0$$

   for any $a, b \in \mathbb{R}$.

   False. E.g. $\displaystyle\int_1^0 1\,dx = x|_1^0 = 0 - 1 = -1.$

   (c) Decide if the following statement is true or false. If true, prove it, if false, give a counterexample. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions, then

$$\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

   False. E.g. if $f(x) = g(x) = 1$ and $a = 0$, $b = 2$, then

$$\int_a^b f(x)g(x)\,dx = \int_0^2 1\,dx = (2-0)1 = 2$$

   and

$$\int_a^b f(x)\,dx \int_a^b g(x)\,dx = \int_0^2 1\,dx \int_0^2 1\,dx = 2 \cdot 2 = 4$$

2. (20 pts) Find the following.
   (a) $\displaystyle\int_{-1}^{8} \frac{x^2}{\sqrt[3]{2x+1}}\,dx$

   Substitute $u = 2x + 1$.

$$\frac{du}{dx} = 2 \implies dx = \frac{du}{2}$$

Also, $x = (u-1)/2$, and the bounds on u are $2(-1)+1 = -1$ and $2(8)+1 = 17$.

$$\begin{aligned}
\int_{-1}^{8} \frac{x^2}{\sqrt[3]{2x+1}}\,dx &= \int_{-1}^{17} \frac{\left(\frac{u-1}{2}\right)^2}{2u^{1/3}}\,du \\
&= \frac{1}{8}\int_{-1}^{17} \frac{u^2-2u+1}{u^{1/3}}\,du = \frac{1}{8}\int_{-1}^{17} u^{5/3} - 2u^{2/3} + u^{-1/3}\,du \\
&= \frac{1}{8}\left(\frac{3}{8}u^{8/3} - 2\frac{3}{5}u^{5/3} - 3u^{2/3}\right)\Big|_{-1}^{17} \\
&= \frac{1}{8}\left(\frac{3}{8}17^{8/3} - \frac{6}{5}17^{5/3} - 3\cdot 17^{2/3} - \frac{3}{8}(-1)^{8/3} + \frac{6}{5}(-1)^{5/3} - 3\cdot(-1)^{2/3}\right) \\
&= \frac{1}{8}\left(\frac{3}{8}17^{8/3} - \frac{6}{5}17^{5/3} - 3\cdot 17^{2/3} - \frac{3}{8} + \frac{6}{5} - 3\right)
\end{aligned}$$

(b) $\displaystyle \frac{d}{dx}\int_{x^3}^{2^x} \ln(t^2+1)\,dt$

Let $F(x) = \int_0^x \ln(t^2+1)\,dt$. By the FTC, $F'(x) = \ln(x^2+1)$. Hence

$$\begin{aligned}
\frac{d}{dx}\int_{x^3}^{2^x} \ln(t^2+1)\,dt &= \frac{d}{dx}\left(\int_{x^3}^{0} \ln(t^2+1)\,dt + \int_0^{2^x} \ln(t^2+1)\,dt\right) \\
&= \frac{d}{dx}\left(-\int_0^{x^3} \ln(t^2+1)\,dt + \int_0^{2^x} \ln(t^2+1)\,dt\right) \\
&= \frac{d}{dx}(F(2^x) - F(x^3)) = F'(2^x)(2^x\ln(2)) - F'(x^3)(3x^2) \\
&= \ln((2^x)^2+1)\,2^x\ln(2) - \ln((x^3)^2+1)\,3x^2 \\
&= \ln(2^{2x}+1)\,2^x\ln(2) - \ln(x^6+1)\,3x^2
\end{aligned}$$

(c) $\displaystyle \int \tan(x)\,dx$. (Hint: $\tan x = \frac{\sin x}{\cos x}$.)

$$\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Substitute $u = \cos x$. Then

$$\frac{du}{dx} = -\sin x \implies \sin(x)dx = -du$$

Hence

$$\int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C$$

(d) $\displaystyle \int_{-2}^{5} \left|x^2-4x+3\right|\,dx$

Notice that $x^2-4x+3 = \pm(x^2-4x+3)$ depending on whether $x^2-4x+3 = (x-1)(x-3)$ is positive or negative. The graph of x^2-4x+3 is a (upright) parabola, so $x^2-4x+3 < 0$

for $1 < x < 3$ and $x^2 - 4x + 3 \geq 0$ otherwise. Therefore

$$\begin{aligned}
\int_{-2}^{5} \left|x^2 - 4x + 3\right| \, dx &= \int_{-2}^{1} x^2 - 4x + 3 \, dx + \int_{1}^{3} -(x^2 - 4x + 3) \, dx + \int_{3}^{5} x^2 - 4x + 3 \, dx \\
&= \int_{-2}^{1} x^2 - 4x + 3 \, dx - \int_{1}^{3} x^2 - 4x + 3 \, dx + \int_{3}^{5} x^2 - 4x + 3 \, dx \\
&= \left(\frac{x^3}{3} - 2x^2 + 3x\right)\Big|_{-2}^{1} - \left(\frac{x^3}{3} - 2x^2 + 3x\right)\Big|_{1}^{3} + \left(\frac{x^3}{3} - 2x^2 + 3x\right)\Big|_{3}^{5} \\
&= \frac{1^3}{3} - 2 \cdot 1^2 + 3 \cdot 1 - \frac{(-2)^3}{3} + 2 \cdot (-2)^2 - 3 \cdot (-2) \\
&\quad - \frac{3^3}{3} + 2 \cdot 3^2 - 3 \cdot 3 + \frac{1^3}{3} - 2 \cdot 1^2 + 3 \cdot 1 \\
&\quad + \frac{5^3}{3} - 2 \cdot 5^2 + 3 \cdot 5 - \frac{3^3}{3} + 2 \cdot 3^2 - 3 \cdot 3 = 26
\end{aligned}$$

3. (15 pts)
   (a) State the definition of the natural log function in terms of an appropriate integral.

   For $x > 0$,

$$\ln(x) = \int_{1}^{x} \frac{1}{t} \, dt$$

   (b) Use the above definition to show that for all $x, y > 0$,

$$\ln(xy) = \ln(x) + \ln(y).$$

   By the FTC,

$$\frac{d}{dx} \ln(x) = \frac{d}{dx} \int_{1}^{x} \frac{1}{t} \, dt = \frac{1}{x}$$

   Notice that

$$\frac{d}{dx} \ln(xy) = \frac{1}{xy} y = \frac{1}{x} = \frac{d}{dx} \ln(x)$$

   This is true for all $x > 0$, that is on the interval $(0, \infty)$. We know that two functions whose derivatives are equal on an interval can only differ by a constant on that interval. Hence

$$\ln(xy) = \ln(x) + C$$

   We can determine C by setting $x = 1$:

$$\ln(y) = \ln(1 \cdot y) = \ln(1) + C = \int_{1}^{1} \frac{1}{t} \, dt + C = 0 + C = C$$

   where the integral is 0 because the upper and lower bounds are the same. We can now conclude

$$\ln(xy) = \ln(x) + \ln(y)$$

4. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\int_{-x}^{x} f(u) \, du = 0$$

   for all $x \geq 0$. Prove that f must be an odd function. (Hint: Differentiate the above integral and use the Fundamental Theorem of Calculus.)

$$\begin{aligned}\frac{d}{dx}\int_{-x}^{x} f(u)\,du &= \frac{d}{dx}\left(\int_{-x}^{0} f(u)\,du + \int_{0}^{x} f(u)\,du\right)\\ &= \frac{d}{dx}\left(-\int_{0}^{-x} f(u)\,du + \int_{0}^{x} f(u)\,du\right) = -f(-x)\frac{d}{dx}(-x) + f(x)\\ &= f(-x) + f(x)\end{aligned}$$

Also

$$\frac{d}{dx}\int_{-x}^{x} f(u)\,du = \frac{d}{dx}0 = 0$$

So for all x, $0 = f(-x) + f(x)$, hence $f(-x) = -f(x)$.

5. (15 pts) **Extra credit problem.** Don't attempt this problem until you are done with everything else.

   Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function.

   (a) Prove that $F(x) = \int_0^x f(t)\,dt$ is even. (Hint: Use the fact that $f(-t) = -f(t)$, then substitute $u = -t$.)

   Note

   $$F(x) = \int_0^x f(t)\,dt = \int_0^x -f(-t)\,dt$$

   Now substitute $u = -t$. This makes $\frac{du}{dt} = -1 \implies dt = -du$, and the integral goes from 0 to $-x$.

   $$F(x) = \int_0^{-x} f(u)\,du = F(-x)$$

   This is true for any x, hence F is an even function.

   (b) Prove that $F(x) = \int_a^x f(t)\,dt$ is even. (Hint: Split $[a, x]$ into two pieces and use the previous result. Or you could use the symmetry of f to observe something about $\int_{-a}^{a} f(t)\,dt$.)

   We know from part (a), that

   $$\int_0^{-x} f(t)\,dt = \int_0^x f(t)\,dt$$

   Therefore

   $$\begin{aligned}F(-x) = \int_a^{-x} f(t)\,dt &= \int_a^0 f(t)\,dt + \int_0^{-x} f(t)\,dt\\ &= \int_a^0 f(t)\,dt + \int_0^x f(t)\,dt = \int_a^x f(t)\,dt = F(x)\end{aligned}$$

   This is again true for all x, hence F is an even function.