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Name: ____________________

Signature: ____________________

Math 1101 Final Exam

Dec 20, 2005

All of your answers must be carefully justified. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. You may refer to any result proved in class, in the text book, or on the homework you turned in, unless otherwise specified. You may leave fractions, $\sqrt{\ }$, e, π, etc. in your answer if you need.

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1. (15 pts) Evaluate the following. Be sure to justify your answers.
   (a) $\lim_{x\to 0^+}\left(\frac{1}{x^2}\right)^x$
   (b) $\lim_{t\to 0}\frac{\cos(t)-1}{t^2}$
   (c) $\frac{d}{dx}\ln\sqrt[x]{\sin^2(x)\cosh(x)}$

2. (30 pts)
   (a) True or false: A strictly decreasing function $f:\mathbb{R}\to\mathbb{R}$ always has an inverse? If true, prove it, if not, give a counterexample.
   (b) Show that
   $$\log_6\left(\frac{99}{20}\right)=2-4\log_6(2)-\log_6(5)+\log_6(11)$$
   (c) Let $f(x)=1/x^2$. Use the definition of the derivative to find $f'(x)$.
   (d) Is $\operatorname{arcsinh}(\sinh(x))=x$ for all $x\in\mathbb{R}$? Justify your answer.
   (e) Use a linear approximation to $f(x)=\sqrt{x}$ at $x_0=100$ to approximate $\sqrt{101}$.
   (f) Use the Fundamental Theorem of Calculus to find
   $$\int_{-\pi/2}^{\pi/2}\cos(2x)\,dx$$
   Plot the graph of $f(x)=\cos(2x)$. Does your result for the integral make sense?

3. (10 pts) Find the equation of the tangent line to the curve
   $$x^y=x^3-3xy-e^3+1$$
   at the point $(e,0)$. As you differentiate, keep in mind that y is a function of x and treat x^y accordingly. (Hint: Is x^y a power function? Is it an exponential function? Is it neither?)

4. (10 pts) The Washington Metropolitan Air Defense Identification Zone (ADIZ) uses a visual warning laser to alert pilots who enter the ADIZ without proper clearance. An alternating red and green laser light is aimed at the cockpit of the unauthorized airplane. Suppose that a small airplane is flying straight and level at an altitude of 6100 ft toward the ground station where the laser is installed. The plane's speed is 100 knots (nautical miles/hour). At what rate in radians/minute does the laser need to turn to stay aimed at the airplane? Simplify your answer as much as you can. You may use your result from Problem 2e or just leave a $\sqrt{\ }$ in it. FYI, $1\,\text{nm}\approx 6100\,\text{ft}$.

5. (10 pts) As the annual Elves vs. Reindeer hockey game is drawing near, Santa realizes that the old ice rink is no longer holding up and he needs to build a new one. Making the ice itself is no problem at the North Pole, but with temperatures hovering around $-110°$F with windchill, he needs to cover it too. He scrapes together the \$31415 left in his account after buying all those toys, and decides on a tent to cover the rink. The tent has the shape of half a cylinder. Its ends are made of corrugated aluminum and the side is of canvas. Fortunately, the Home Depot's winter 3-for-the-price-of-1 sale includes corrugated aluminum at \$4 for $3\,\text{ft}^2$ and canvas at \$2 for $3\,\text{ft}^2$. Santa wants to maximize the volume of the tent. What are the dimensions of the biggest tent he can afford to build?

6. (10 pts) Use Rolle's Theorem and mathematical induction to prove that a polynomial of degree n has at most n real roots.

7. (20 pts) **Extra credit problem.** We will prove Cauchy's Mean Value Theorem, which is a generalized version of the MVT. It says that if f and g are functions that are continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) \neq 0$ for any $x \in (a, b)$, then there exists a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(b)}{g(a) - g(b)}$$

   (a) Our first attempt to prove this might be as follows. Given the above conditions on f, the MVT says there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

   Similarly, the MVT says that there exists a $c \in (a, b)$ such that

$$g'(c) = \frac{g(a) - g(b)}{a - b}$$

   Now divide these last two equations to get

$$\frac{f'(c)}{g'(c)} = \frac{\frac{f(a)-f(b)}{a-b}}{\frac{g(a)-g(b)}{a-b}} = \frac{f(a) - f(b)}{g(a) - g(b)}$$

   Find the mistake in this argument.

   (b) Recall how the proof of the MVT went. We let

$$y = \frac{f(a) - f(b)}{a - b}(x - a) + f(a),$$

   which is the equation of the secant line to f between a and b. Then we let

$$h(x) = f(x) - y = f(x) - \frac{f(a) - f(b)}{a - b}(x - a) - f(a)$$

   and used Rolle's Theorem on h to claim that c exists. Now we will let

$$h(x) = f(x) - \frac{f(a) - f(b)}{g(a) - g(b)}(g(x) - g(a)) - f(a)$$

   instead. Verify that h satisfies the three conditions of Rolle's Theorem.

   (c) Now use Rolle's Theorem on h to find a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(b)}{g(a) - g(b)}$$