MATH 1101 EXAM 3 SOLUTIONS

Apr 7, 2006

1. (10 pts)

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at a. Prove that f is continuous at a.

   Since f is differentiable at a,

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   exists. Hence

   $$\begin{aligned} \lim_{x \to a}(f(x) - f(a)) &= \lim_{x \to a}(f(x) - f(a))\frac{x - a}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right)\left(\lim_{x \to a} x - a\right) = f'(a) \cdot 0 = 0 \end{aligned}$$

   This shows

   $$0 = \lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a} f(x) - \lim_{x \to a} f(a) = \lim_{x \to a} f(x) - f(a)$$

   Hence

   $$\lim_{x \to a} f(x) = f(a)$$

   (b) Now suppose $f : \mathbb{R} \to \mathbb{R}$ is a function that is continuous at a. Does f have to be differentiable at a? If yes, give a proof, if not give a counterexample and carefully justify it.

   A continuous function need not be differentiable. For example, $f(x) = |x|$ is continuous at 0 but not differentiable there. It is continuous because

   $$\begin{aligned} \lim_{x \to 0^-} |x| &= \lim_{x \to 0^-} -x = 0 \\ \lim_{x \to 0^+} |x| &= \lim_{x \to 0^-} x = 0 \end{aligned}$$

   and so

   $$\lim_{x \to 0^+} |x| = |0|$$

   But it is not differentiable because

   $$\begin{aligned} \lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} &= \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1 \\ \lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} &= \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{-x}{x} = 1 \end{aligned}$$

   and so the derivative

   $$\lim_{x \to 0} \frac{|x| - |0|}{x - 0}$$

   does not exist.

2. (10 pts each)

   (a) Find

   $$\lim_{y \to 0} \frac{y}{\tan(y)}$$

   You may use any result proved in class or in the textbook. Don't use l'Hospital's rule since we haven't covered it.

$$\lim_{y\to 0}\frac{y}{\tan(y)} = \lim_{y\to 0}\frac{y\cos(y)}{\sin(y)} = \left(\lim_{y\to 0}\frac{y}{\sin(y)}\right)\left(\lim_{y\to 0}\cos(y)\right)$$

The first limit exists and is 1 because we know

$$\lim_{y\to 0}\frac{\sin(y)}{y} = 1$$

The second limit exists because cos is continuous at 0, hence

$$\lim y \to 0 \cos(y) = \cos(0) = 1$$

So

$$\lim_{y\to 0}\frac{y}{\tan(y)} = 1$$

(b) Find $\frac{dy}{dx}$ for the implicit function $e^{x^2+y^2} = y\tan(y)\sin(x)$.

$$\begin{aligned}
\frac{d}{dx}e^{x^2+y^2} &= \frac{d}{dx}y\tan(y)\sin(x)\\
e^{x^2+y^2}(2x+2y\frac{dy}{dx}) &= \left(\frac{dy}{dx}\tan(y)+y\sec^2(y)\frac{dy}{dx}\right)\sin(x)+y\tan(y)\cos(x)\\
2xe^{x^2+y^2}-y\tan(y)\cos(x) &= \frac{dy}{dx}(\tan(y)+y\sec^2(y)-2ye^{x^2+y^2})\\
\frac{dy}{dx} &= \frac{\tan(y)+y\sec^2(y)-2ye^{x^2+y^2}}{2xe^{x^2+y^2}-y\tan(y)\cos(x)}
\end{aligned}$$

Or we could have used logarithmic differentiation or the generalized product rule:

$$\frac{d}{dx}(fgh) = (f'g+fg')h+fgh' = f'gh+fg'h+fgh'$$

(c) Find the equation of the tangent line to

$$f(x) = \frac{\ln(2/x)}{\sqrt[3]{12-x^2}}$$

at $x = 2$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\frac{\ln(2)-\ln(x)}{\sqrt[3]{12-x^2}} = \frac{-\frac{1}{x}\sqrt[3]{12-x^2}-\ln\left(\frac{2}{x}\right)\frac{1}{3}(12-x^2)^{-2/3}(-2x)}{\sqrt[3]{12-x^2}^2}\\
f'(2) &= \frac{-\frac{1}{2}\sqrt[3]{8}-\ln(1)\frac{1}{3}(8)^{-2/3}(-4)}{\sqrt[3]{8}^2} = -\frac{1}{4}
\end{aligned}$$

Also

$$f(2) = \frac{\ln(2/2)}{\sqrt[3]{8}} = 0$$

So the equation of the tangent line is

$$\begin{aligned}
y-0 &= -\frac{1}{4}(x-2)\\
y &= \frac{1}{2}-\frac{x}{4}
\end{aligned}$$

3. (10 pts)
   (a) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable at the point $a \in \mathbb{R}$.

   f is differentiable at a if the derivative

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   exists.

   (b) Let $f(x) = x^2$. Use the definition of the derivative to find $f'(x)$.

   $$\frac{d}{dx} x^2 = \lim_{t \to x} \frac{t^2 - x^2}{t - x} = \lim_{t \to x} \frac{(t - x)(t + x)}{t - x} = \lim_{t \to x} (t + x) = 2x$$

4. (10 pts) Use the fact that $\frac{d}{dx} \cos(x) = -\sin(x)$ to find the derivative of $\arccos(x)$. Simplify your answer as much as possible.

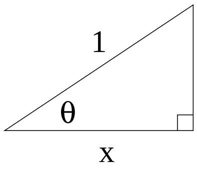
$$1 = \frac{d}{dx} x = \frac{d}{dx} \cos(\arccos(x)) = -\sin(\arccos(x)) \frac{d}{dx} \arccos(x)$$

$$\implies \frac{d}{dx} \arccos(x) = -\frac{1}{\sin(\arccos(x))}$$

Look at the triangle on the right and observe

$$x = \cos(\theta) \Leftrightarrow \arccos(x) = \theta$$

$$\sin(\arccos(x)) = \sin(\theta) = \sqrt{1 - x^2}$$



Hence

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1 - x^2}}$$

5. **Extra credit problem.** Don't attempt this problem until you are done with everything else.

   Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

   (a) (5 pts) Use the chain rule to prove that f' is an even function.

   Since f is odd, we know $f(x) = -f(-x)$. So

   $$f'(x) = \frac{d}{dx} - f(-x) = -f'(-x)(-1) = -f'(x)$$

   (b) (10 pts) Use the definition of the derivative to prove that f' is an even function.

   $$f'(-x) = \lim_{t \to -x} \frac{f(t) - f(-x)}{t - (-x)} = \lim_{t \to -x} \frac{f(t) + f(x)}{t + x}$$

   Now, let's substitute $s = -t$. Notice that as $t \to -x$, $s \to x$, so

   $$\lim_{t \to -x} \frac{f(t) + f(x)}{t - x} = \lim_{s \to x} \frac{f(-s) + f(x)}{-s + x} = \lim_{s \to x} \frac{-f(s) + f(x)}{-s + x} = f'(x)$$

   So $f'(-x) = f'(x)$ for all x.