Math 1101 Exam 1 Solutions

Jan 27, 2006

1. (9 pts)
   (a) Define what a function is.

   A function from a set S to a set T is a rule that sends every element of S to a unique element of T.

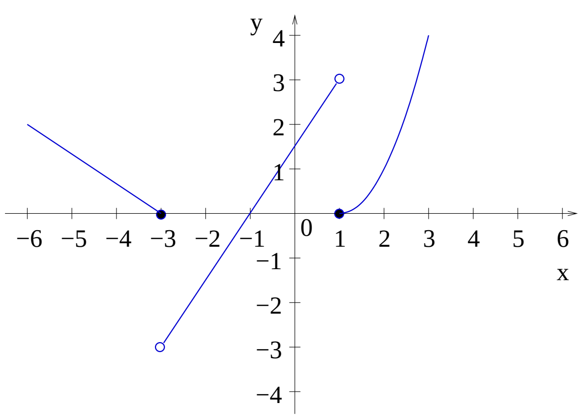
   (b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be even.

   A function $f : \mathbb{R} \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

   (c) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be strictly increasing.

   A function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing if $f(x_1) < f(x_2)$ for all $x_1 < x_2 \in R$.

2. (10 pts) Below is the graph of a piecewise defined function. Write a formula for the function.



You can write down the equations of the line segments by determining the slope first as $\Delta y/\Delta x$, then substituting any point on the line segment. E.g. for the leftmost one, y decreases by 2 as x goes from -6 to -3, so the slope is $-2/3$. Now substitute $(-3, 0)$ into $y = -2/3\,x + b$ to get $b = -2$. Notice that the half parabola is the graph of $f(x) = x^2$ for $x \geq 0$ shifted to the right by 1.

$$f(x) = \begin{cases} -\frac{2}{3}x - 2 & \text{if } x \leq -3 \\ \frac{3}{2}x + \frac{3}{2} & \text{if } -3 < x < 1 \\ (x-1)^2 & \text{if } 1 \leq x \end{cases}$$

3. (21 pts) Decide if the following statements are true or false and justify your answer. (Remember that you can show a statement is false by providing a counterexample, but you cannot prove a statement true by giving an example.)
   (a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both increasing functions, so is fg.

   False. E.g. $f(x) = x$ and $g(x) = x$ are both increasing functions, but $fg(x) = x^2$ is not.

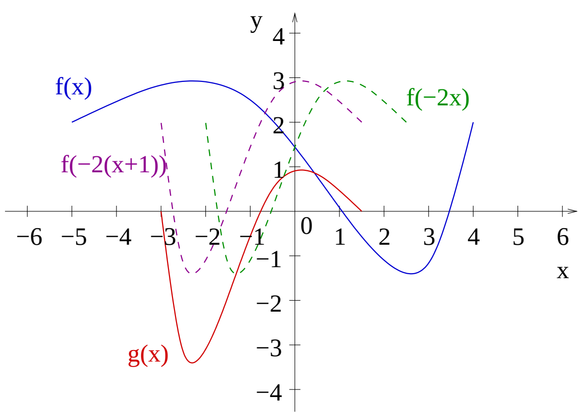
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is an odd function and $g : \mathbb{R} \to \mathbb{R}$ is an even function, then $f \circ g$ is an odd function.

   False. E.g. $f(x) = x$ is an odd function, $g(x) = x^2$ is even, and $f \circ g(x) = x^2$ is clearly not an odd function.

   (c) If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then f cannot be even.

True. Let $x > 0$. Then $-x < 0$, so $-x < x$, hence $f(-x) < f(x)$ by the strictly increasing property. Hence $f(-x) \neq f(x)$ and therefore f cannot be even.

4. (10 pts) The graph of $g(x)$ given below can be obtained from the graph of $f(x)$–also given below–by a sequence of transformations, such as shifts, stretches/compressions, and reflections. If $g(x) = Cf(Ax + B) + D$, find A, B, C, D.



Notice the graph was reflected about the y-axis, compressed by a factor of 2 in the horizontal direction, shifted left by 1 unit and down by 2 units. These correspond respectively to multiplying the input by -2, adding 1 to it, and subtracting 2 from f. So $g(x) = f(-2(x+1)) - 2 = f(-2x - 2) - 2$. (Remember that multiplication is done before addition, so when you add 1 to x, you need to use parentheses.)

So $A = B = D = -2$ and $C = 1$.

5. (10 pts) Let $f(x) = \sqrt{\frac{(x+1)(x-2)}{x+5}}$. What is the largest possible subset S of the real numbers that could be the domain of f?

Since we are dividing by $x + 5$, $x \neq 5$, otherwise we would be dividing by 0. Because of the square root, we need

$$\frac{(x + 1)(x - 2)}{x + 5} \geq 0$$

Notice

$$x - 2 < x + 1 < x + 5$$

We need either all three of these to be nonnegative (actually, $x + 5$ cannot be 0, so it has to be positive), or two of them to be nonpositive and the third one to be nonnegative. Because of the order above, the former will happen if and only if $0 \leq x - 2$ because this will force $0 < x + 1 < x + 5$. The latter will happen if and only if

$$x - 2 < x + 1 \leq 0 < x + 5$$

which is true when $x \leq -1$ and $-5 < x$. So the largest possible domain is $(-5, -1] \cup [2, \infty)$.