GMS 91 Exam 2 Solutions

Nov 12, 2008

1. (5 pts) Simplify the expression

$$\frac{1}{x+3} - \frac{1}{x+6}$$

and give your answer in the form

$$\frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are some expressions in x.

$$\frac{1}{x+3} - \frac{1}{x+6} = \frac{(x+6)-(x+3)}{(x+3)(x+6)} = \frac{3}{(x+3)(x+6)}.$$

2. (10 pts) Solve the following equation.

$$\frac{1}{|2x-3|} = 5$$

First, note that $2x-3$ cannot be 0, so $|2x-3| \neq 0$. Now multiply both sides by $|2x-3|$:

$$1 = 5|2x-3|$$
$$\frac{1}{5} = |2x-3|$$

This means $2x-3$ is either $1/5$ or $-1/5$.

Case $2x-3=1/5$**:**

$$2x-3 = \frac{1}{5}$$
$$2x = \frac{1}{5}+3 = \frac{16}{5}$$
$$x = \frac{8}{5}$$

Case $2x-3=-1/5$**:**

$$2x-3 = -\frac{1}{5}$$
$$2x = -\frac{1}{5}+3 = \frac{14}{5}$$
$$x = \frac{7}{5}$$

Hence the solution is $x = 7/5$ or $x = 8/5$.

I will check my answer for good measure:

$$\frac{1}{|2\frac{8}{5}-3|} = \frac{1}{|\frac{16-15}{5}|} = \frac{1}{|\frac{1}{5}|} = \frac{1}{\frac{1}{5}} = 5 \qquad \surd$$

$$\frac{1}{|2\frac{7}{5}-3|} = \frac{1}{|\frac{14-15}{5}|} = \frac{1}{|-\frac{1}{5}|} = \frac{1}{\frac{1}{5}} = 5 \qquad \surd$$

3. (15 pts) Solve the following equation.

$$||5x+3|-13| = 1$$

$||5x+3|-13| = 1$ means that $|5x+3|-13$ is 1 unit away from 0 on the number line. That is either $|5x+3|-13 = 1$ or $|5x+3|-13 = -1$.

Case $|5x+3|-13=1$**:**

$$|5x+3|-13 = 1 \implies |5x+3| = 14$$

which means either $5x+3 = 14$ or $5x+3 = -14$.

$$5x+3 = 14 \implies 5x = 11 \implies x = \frac{11}{5}$$

$$5x+3 = -14 \implies 5x = -17 \implies x = -\frac{17}{5}$$

So we get two solutions out of this case $x = 11/5$ or $x = -17/5$.

Case $|5x+3|-13=-1$**:**

$$|5x+3|-13 = -1 \implies |5x+3| = 12$$

which means either $5x+3 = 12$ or $5x+3 = -12$.

$$5x+3 = 12 \implies 5x = 9 \implies x = \frac{9}{5}$$

$$5x+3 = -12 \implies 5x = -15 \implies x = -3$$

So we get two solutions out of this case $x = 9/5$ or $x = -3$.

So the solution of this equation is $x = 11/5$ or $x = -17/5$ or $x = 9/5$ or $x = -3$.

I will again check my answer:

$$\left|\left|5\frac{11}{5}+3\right|-13\right| = ||11+3|-13| = ||14|-13| = |14-13| = |1| = 1 \qquad \surd$$

$$\left|\left|5\left(-\frac{17}{5}\right)+3\right|-13\right| = ||-17+3|-13| = ||-14|-13| = |14-13| = |1| = 1 \qquad \surd$$

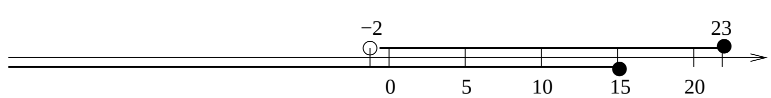
$$\left|\left|5\frac{9}{5}+3\right|-13\right| = ||9+3|-13| = ||12|-13| = |12-13| = |-1| = 1 \qquad \surd$$

$$||5\cdot(-3)+3|-13| = ||-15+3|-13| = ||-12|-13| = |12-13| = |-1| = 1 \qquad \surd$$

4. (6 pts) Let $S = (-\infty, 15]$ and $T = (-2, 23]$.

(a) Find $S \cup T$.

Here is the graph of these intervals on the real line:



This shows $S \cup T = (\infty, 23]$.

(b) Find $S \cap T$.

Refer to the graph above again to see $S \cap T = (-2, 15]$.

5. (14 pts)

(a) We learned that multiplication on the real numbers is associative. Explain what this means and give an example.

It means that if x, y, z are any real numbers, then $(xy)z = x(yz)$. For example

$$(2 \cdot 3)4 = 6 \cdot 4 = 24 \text{ and } 2(3 \cdot 4) = 2 \cdot 12 = 24.$$

(b) Use the usual principles (associativity of addition, commutativity of multiplication, distributivity of multiplication over addition, etc) of real number arithmetic to multiply out

$$(2x - y)(x^2 + 7y + 3).$$

At each step, explain which particular principle you are using.

$$\begin{aligned}
&(2x - y)(x^2 + 7y + 3) \\
&= (2x + (-y))(x^2 + 7y + 3) && \text{by additive inverse property} \\
&= 2x(x^2 + 7y + 3) + (-y)(x^2 + 7y + 3) && \text{by distributivity of } \cdot \text{ over } + \\
&= \big((2x)(x^2) + (2x)(7y) + (2x)3\big) && \text{by distributivity of } \cdot \text{ over } + \\
&\quad + \big((-y)(x^2) + (-y)(7y) + (-y)(3)\big) \\
&= (2x)(x^2) + (2x)(7y) + (2x)3 && \text{by associativity of } + \\
&\quad + (-y)(x^2) + (-y)(7y) + (-y)(3) \\
&= 2x^3 + 14xy + 6x && \text{by associativity and} \\
&\quad + (-yx^2) + (-7y^2) + (-3y) && \text{commutativity of } \cdot \\
&= 2x^3 + 14xy + 6x - yx^2 - 7y^2 - 3y && \text{by additive inverse property}
\end{aligned}$$

Here is one example how associativity and commutativity of multiplication is used:

$$\begin{aligned}
(2x)(7y) &= 2(x7)y && \text{by associtivity of } \cdot \\
&= 2(7x)y && \text{by commutativity of } \cdot \\
&= (2 \cdot 7)(xy) && \text{by associtivity of } \cdot \\
&= 14xy && \text{by associtivity of } \cdot
\end{aligned}$$

6. (10 pts) Solve the inequality

$$\frac{y - 5}{y + 20} \geq 3$$

and give your answer both in interval and set builder notations.

First, note that $y + 20 \neq 0$ because we can't divide by 0. We will want to mupltiply both sides by $y + 20$, but this could flip the direction of the inequality. So we need to consider two cases:

Case $y + 20 > 0$: In this case, we get

$$\begin{aligned}
y - 5 &\geq 3(y + 20) \\
y - 5 &\geq 3y + 60 \\
-65 &\geq 2y \\
y &\leq -\frac{65}{2}
\end{aligned}$$

The condition $y+20>0$ tells us $y>-20$. Since $-20>-\frac{65}{2}$, there is no number y such that $-20<y$ and $y\leq -\frac{65}{2}$. So we get no solutions in this case.

Case $y+20<0$: In this case, we get

$$\begin{aligned}
y-5 &\leq 3(y+20)\\
y-5 &\leq 3y+60\\
-65 &\leq 2y\\
y &\geq -\frac{65}{2}
\end{aligned}$$

The condition $y+20<0$ tells us $y<-20$. So the solution we get out of this case is $-\frac{65}{2}\leq y<-20$.

In interval notation, the solution is $\left[-\frac{65}{2},-20\right)$. In set builder notation, this is $\left\{y\in\mathbb{R}\mid -\frac{65}{2}\leq y<-20\right\}$.

I will check a few numbers from this range:

$y=-30$**:**

$$\frac{-30-5}{-30+20}=\frac{-35}{-10}=\frac{7}{2}\geq 3 \qquad \surd$$

$y=-21$**:**

$$\frac{-21-5}{-21+20}=\frac{-26}{-1}=26\geq 3 \qquad \surd$$

7. (10 pts) **Extra credit problem.** Solve the inequality

$$\frac{x-2}{(x+5)^2}\leq\frac{7}{x+8}.$$

First, note that $x+5\neq 0$ and $x+8\neq 0$, otherwise we would be dividing by 0. We will multiply both sides of the inequality by $(x+5)^2$. The good thing is that $(x+5)^2\geq 0$, so there is no danger that the direction of the inequality flips as a consequence. We now have

$$x-2\leq\frac{7(x+5)^2}{x+8}.$$

Next, we multiply this by $x+8$. This could be positive or negative, so the direction of the inequality could flip. So we need to consider two cases.

Case $x+8>0$: In this case, we get

$$\begin{aligned}
(x-2)(x+8) &\leq 7(x+5)^2\\
x^2+6x-16 &\leq 7(x^2+10x+25)=7x^2+70x+175\\
0 &\leq 6x^2+64x+191
\end{aligned}$$

Next, we need to factor $6x^2+64x+191$. We can do this by completing the square

$$\begin{aligned}
6x^2+64x+191 &= 6\left(x^2+\frac{32}{3}x+\frac{191}{6}\right)=6\left(\left(x+\frac{16}{3}\right)^2-\frac{16^2}{3^2}+\frac{191}{6}\right)\\
&=6\left(\left(x+\frac{16}{3}\right)^2-\frac{2\cdot 256-3\cdot 191}{18}\right)=6\left(\left(x+\frac{16}{3}\right)^2-\frac{-61}{18}\right)\\
&=6\left(\left(x+\frac{16}{3}\right)^2+\frac{61}{18}\right)
\end{aligned}$$

Notice that $(x + 16/3)^2 \geq 0$, since it's a square, the right hand side above is always positive, for any value of x. Therefore any real number which satisfies $x + 8 > 0$ solves the original inequality. This gives us $x > -8$.

Case $x + 8 < 0$**:** In this case, we get

$$\begin{aligned} (x-2)(x+8) &\geq 7(x+5)^2 \\ x^2 + 6x - 16 &\geq 7(x^2 + 10x + 25) = 7x^2 + 70x + 175 \\ 0 &\geq 6x^2 + 64x + 191 \end{aligned}$$

We saw in the previous case that the RHS of the above inequality is always positive. Therefore no real number x satisfies the inequality in this case.

We conclude that the solution of the inequality is $x > -8$. But we noted that $x + 5$ cannot be 0, so we need to exclude $x = -5$ from this interval. The final result is $x \in (-8, -5) \cup (-5, \infty)$.

I will check a few numbers from this range:

$x = -6$**:**

$$\frac{-6-2}{(-6+5)^2} = \frac{-8}{(-1)^2} = -8, \qquad \frac{7}{-6+8} = \frac{7}{2} \text{ and } -8 < \frac{7}{2} \qquad \checkmark$$

$x = -4$**:**

$$\frac{-4-2}{(-4+5)^2} = \frac{-6}{(1)^2} = -6, \qquad \frac{7}{-4+8} = \frac{7}{4} \text{ and } -8 < \frac{7}{4} \qquad \checkmark$$

$x = 2$**:**

$$\frac{2-2}{(2+5)^2} = 0, \qquad \frac{7}{2+8} = \frac{7}{10} \text{ and } 0 < \frac{7}{10} \qquad \checkmark$$