Set Properties

Some sets

A $= \{$a, b, c$\}$

B $= \{1, 2, 3\}$

B' $= \{1, 2\}$

C $= \{$ The Amazon River,
Bill Clinton's left eyebrow,
$3\}$

These sets have all been specified by a simple means: **Listing**

Set Membership

The relation between a set and one of its members is called the **set membership** relation. It is notated like this:

$$a \in A$$

When we want to say the opposite, we use $\notin$

$$\text{Bill Clinton} \notin A$$

Cardinality

$A = \{$a, b, c$\}$

$B = \{1, 2, 3\}$

$B' = \{1, 2\}$

$C = \{\text{The Amazon}, \text{BillClinton's lefteyebrow}, 3\}$

The size of a set A is called its **cardinality**, written $|A|$:

$|A| =$

$|B| =$

$|C| =$

Which of the following are true?

$|A| \in A$

$|A| \in B$

$|B| \in B$

$|B| = |A|$

Cardinality

$\mathrm{A} = \{$a, b, c$\}$

$\mathrm{B} = \{1, 2, 3\}$

$\mathrm{B}' = \{1, 2\}$

$\mathrm{C} = \{\ \mathrm{The\ Amazon}, \mathrm{BillClinton's\ lefteyebrow}, 3\}$

The size of a set A is called its **cardinality**, written $|\mathrm{A}|$:

$$|\mathrm{A}| = 3$$
$$|\mathrm{B}| =$$
$$|\mathrm{C}| =$$

Which of the following are true?

$$|\mathrm{A}| \in \mathrm{A}$$
$$|\mathrm{A}| \in \mathrm{B}$$
$$|\mathrm{B}| \in \mathrm{B}$$
$$|\mathrm{B}| = |\mathrm{A}|$$

Cardinality

$A = \{$a, b, c$\}$

$B = \{1, 2, 3\}$

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$$|B| = |A|$$

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$$|A| = 3$$

$$|B| = 3$$

$$|C| = 3$$

Which of the following are true?

$|A| \in A$ False

$|A| \in B$

$|B| \in B$

$|B| = |A|$

Cardinality

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$|A| \in B$ True

$|B| \in B$

$|B| = |A|$

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Sets by predicates

The general idea is that we accept any way of specifying a set that gives us a definite way of knowing if something is in the set or not.

The set of all odd numbers:

$$\mathbf{OddSet} = \{x \mid \text{Odd}(x)\}$$

Subset Relation

We say

$$A \subseteq B$$

("A is a subset of B") if and only if every member of A is a member of B.

In the examples above:

$A = \{$a, b, c$\}$

$B = \{1, 2, 3\}$

$B' = \{1, 2\}$

$C = \{\ \text{The Amazon}, \text{Bill Clinton's left eyebrow}, 3\}$

$$B' \subseteq B$$

$$A \not\subseteq B$$

Read this:

B' is a subset of B

$A$is not a subset of B

Universe of Discourse and Powerset

Universe of Discourse

Let

$$\aleph = \{1, 2, 3, \ldots\}$$

$\aleph$ is called the set of **Natural Numbers.**
In arithmetic we call $\aleph$ the **universe of discourse**, the set of things we're talking about. This is sometimes called the set of **atoms**. The term atom contrasts with the term set. In set theory we talk mostly about sets, but the things we talk about that *aren't* sets are atoms. Atoms don't have members.

Why have one?

Being clear about the universe of discourse makes our set definitions clearer:

$$\mathbf{OddSet} = \{x \mid x \in \aleph \text{ and } \mathrm{Odd}(x)\}$$

This makes it clear that we’re not including negative numbers in oddset.

Universe of sets

Along with the idea of a universe of discourse $\mathcal{U}$ comes the idea of a universe of sets **generated** from $\mathcal{U}$:

(1) $\mathcal{U} = \{\mathrm{a, b, c}\}$

Some sets in the universe of sets of $\mathcal{U}$

$\emptyset$	0 element sets
{a}, {b}, {c}	1 element sets
{a,b}, {b,c}, {a,c}	2 element sets
{a,b,c}	3 element sets

More sets still!

Some more sets in the universe of sets of $\mathcal{U}$

$$\{\emptyset\}$$
$$\{\{\emptyset\}\}$$
$$\vdots$$
$$\{\{\text{a}\}\}$$
$$\{\{\{\text{a}\}\}\}$$
$$\vdots$$
$$\{\{\text{a,b}\}\}$$
$$\{\{\{\text{a,b}\}\}\}$$
$$\vdots$$
$$\{\{\text{a,b,c}\}\}$$
$$\vdots$$

Powerset

What we did in (1) was list all the subsets of $\mathcal{U}$. That gives a colection of sets. A set of sets. This new big set is called the powerset:

(2) $\mathcal{U} = \{\text{a, b, c}\}$

$$\wp(\mathcal{U}) = \{\emptyset, \{\text{a}\}, \{\text{b}\}, \{\text{c}\}$$
$$\{\text{a,b}\}, \{\text{b,c}\}, \{\text{a,c}\}$$
$$\{\text{a,b,c}\}\ \}$$

Infinite set collections

The universe generated by a particular universe of discourse is not only BIGGER than the power set

(3) $\mathcal{U} = \{\text{a, b, c}\}$

Sets 1 element sets in the universe of sets of $\mathcal{U}$

$\{\text{a}\}, \{\{\text{a}\}\}, \{\{\{\text{a}\}\}\}, \ldots$ 1 element sets

The universe of sets is infinite. The set of 1-element sets in the universe of sets is infinite.

Membership/cardinality revisited

$a \in \{a, b, c\}$

$a \in \{\{a, b, c\}\}$

$|\{\{a, b, c\}\}| = 3$

$\{a, b, c\} \in \{\{a, b, c\}\}$

$|\{\{a, b, c\}\}| = 1$

Membership/cardinality revisited

$a \in \{a, b, c\}$ True

$a \in \{\{a, b, c\}\}$

$|\{\{a, b, c\}\}| = 3$

$\{a, b, c\} \in \{\{a, b, c\}\}$

$|\{\{a, b, c\}\}| = 1$

Membership/cardinality revisited

a $\in \{$a, b, c$\}$	True
a $\in \{\{$a, b, c$\}\}$	False
$\mid\{\{$a, b, c$\}\}\mid = 3$	
$\{$a, b, c$\} \in \{\{$a, b, c$\}\}$	
$\mid\{\{$a, b, c$\}\}\mid = 1$	

Membership/cardinality revisited

$a \in \{a, b, c\}$	True
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Membership/cardinality revisited

$a \in \{a, b, c\}$	True
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$\{a, b, c\} \in \{\{a, b, c\}\}$	True
$\|\{\{a, b, c\}\}\| = 1$	True

Summary Thus far

1. A set is a collection of things. The things in a set are called its members.
2. Anything can be a member of a set, including another set (and Bill Clinton's eyebrow!).
3. A set X that contains only members of another set Y is called a **subset** of y.
4. The empty set is a subset of every set.
5. Sets can be both subsets and members of other sets. A $\subseteq$ B and A $\in$ B.

$$\text{A} = \{\text{a}, \text{b}\}$$
$$\text{B} = \{\{\text{a}, \text{b}\}, \text{a}, \text{b}\}$$

Set Union

$$\text{A} = \{\text{a,b, c}\}$$
$$\text{B} = \{1,\ 2,\ 3\}$$

Set Union of A and B:

$$\text{A} \bigcup \text{B} = \{\text{a,b, c,}\ 1,\ 2,\ 3\}$$

The union of two sets is a set that contains everything that's in **either** set.

How many such sets are there?

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How many such sets are there?

$$\{\text{a,b, c,} 1, 2, 3, \text{ Bill Clinton's eyebrow}\}$$

Set Union: Improved definition

Set Union of A and B:

$$\text{A} \bigcup \text{B} = \{\text{a,b, c,} 1, 2, 3\}$$

The union of two sets is the smallest set that contains everything that's in **either** set.

$$\text{A} \bigcup \text{B} \neq \{\text{a,b, c,} 1, 2, 3, \text{ Bill Clinton's eyebrow}\}$$

Set Intersection

Set intersection of B and B':

$$A = \{a, b, c\}$$
$$B = \{1, 2, 3\}$$
$$B' = \{1, 2\}$$
$$B' \bigcap B = \{1, 2\}$$

The intersection of two sets is a set that contains only things that are in **both** sets.
How many such sets are there for B, B'?

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The intersection of two sets is the largest set that contains only things that are in **both** sets.
How many such sets are there for B, B'?

$$\{1\}$$

The Empty Set I

What set is:

$$\mathsf{A} \bigcap \mathsf{B} = \{?\}$$

There is no element that is shared by A and B.
But by definition there has to be a set X such that:

$$\mathsf{A} \bigcap \mathsf{B} = \mathsf{X}$$

So we make up a set that fits the definition. It's called the empty set and is written $\emptyset$.

The Empty Set II

The empty set has no members:

$$\emptyset = \{\,\}$$

The empty set is a subset of all sets:

$$\emptyset \subseteq \mathsf{A}$$

We said there is always an infinite number of sets generated by any universe of discourse. That's true even when $\mathcal{U}$ equals the empty set. Why?

The Empty Set II

The empty set has no members:

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$$\{\emptyset\},\ \{\{\emptyset\}\},\ \ldots$$

Complement

We will write the complement of a set B as B′:
The complement of a set is everything in the universe of discourse that is not in the set:

$$\mathcal{U} = \{a,\ b,\ c\}$$
$$\mathsf{B} = \{\ b,\ c\}$$
$$\mathsf{B}' = \{a\}$$

$$\mathsf{B} \bigcup \mathsf{B}' = ?$$
$$\mathsf{B} \bigcap \mathsf{B}' = ?$$

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The complement of a set is everything in the universe of discourse that is not in the set:

$$\mathcal{U} = \{a,\ b,\ c\}$$
$$\mathsf{B} = \{\ b,\ c\}$$
$$\mathsf{B}' = \{a\}$$

$$\mathsf{B} \bigcup \mathsf{B}' = \mathcal{U}$$
$$\mathsf{B} \bigcap \mathsf{B}' = ?$$

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The complement of a set is everything in the universe of discourse that is not in the set:

$$\mathcal{U} = \{a,\ b,\ c\}$$
$$\mathsf{B} = \{\,b,\ c\,\}$$
$$\mathsf{B}' = \{a\,\}$$

$$\mathsf{B} \bigcup \mathsf{B}' = \mathcal{U}$$
$$\mathsf{B} \bigcap \mathsf{B}' = \emptyset$$

Complement question

In general is it the case that

$$A' \subseteq A?$$

Is it ever the case?

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When $A = \mathcal{U}$:

$$\mathcal{U}' = ?$$

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When $A = \mathcal{U}$:

$$\mathcal{U}' = \emptyset$$

$$\emptyset \subseteq \mathcal{U} \quad ?$$

Complement question

In general is it the case that

$$A' \subseteq A?$$

Is it ever the case?

When $A = \mathcal{U}$:

$$\mathcal{U}' = \emptyset$$

$$\emptyset \subseteq \mathcal{U} \quad \text{True}$$