Partially ordered sets and lattices

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Preliminaries

We define an **order** as a transitive relation R that can be either

1. reflexive and antisymmetric (**a weak partial order**), written

$$\leq, \subseteq, \sqsubseteq, \ldots$$

2. irreflexive and asymmetric (**a strong partial order**), written

$$<, \subset, \sqsubset, \langle\!\langle, \ldots$$

Posets

Definition 1. *A poset (***p***artially-***o***rdered set) is a set together with a weak order.*

Examples: Weak Orders

Examples of weak orders are $\subseteq$ on sets and $\leq$ on number. The case of $\subseteq$:

Transitive	$\dfrac{A \subseteq B \quad B \subseteq C}{A \subseteq C}$
Reflexive	$A \subseteq A$
Antisymmetric	$\dfrac{A \subseteq B \quad B \subseteq A}{A = B}$

Examples: Strong Orders

An English example of a strong order is *ancestor-of*.

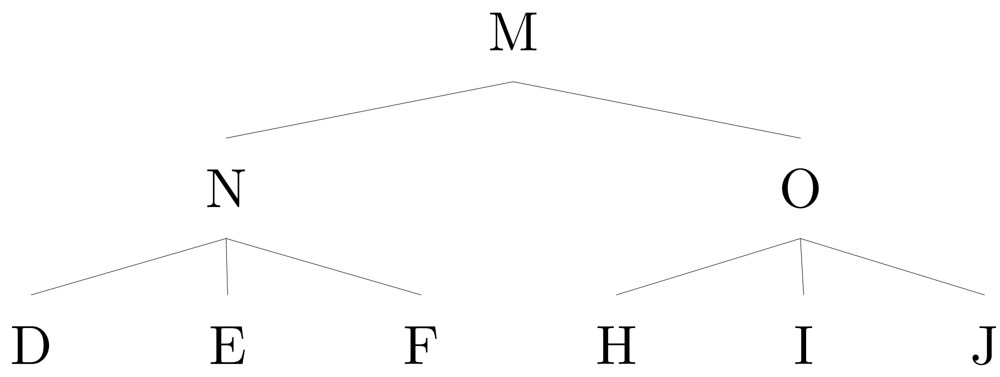
Transitive	A is the ancestor of B B is the ancestor of C ——— A is the ancestor of C
Irreflexive	A is not an ancestor of A
Asymmetric	A is the ancestor of B ——— B is not the ancestor of A

More beautiful than

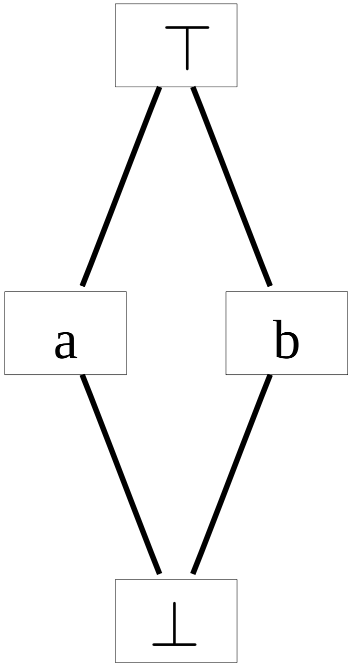
Transitive	Lassie is more beautiful than Rin Tin Tin Rin Tin Tin is more beautiful than Asta ――――――――――――― Lassie is more beautiful than Asta
Irreflexive	Lassie is more beautiful than Lassie
Asymmetric	Lassie is more beautiful than Fido ――――――――――――― Fido is not more beautiful than Lassie

More Examples

1. *is taller than*
2. *is 2 inches taller than*?
3. The dominates relation in trees

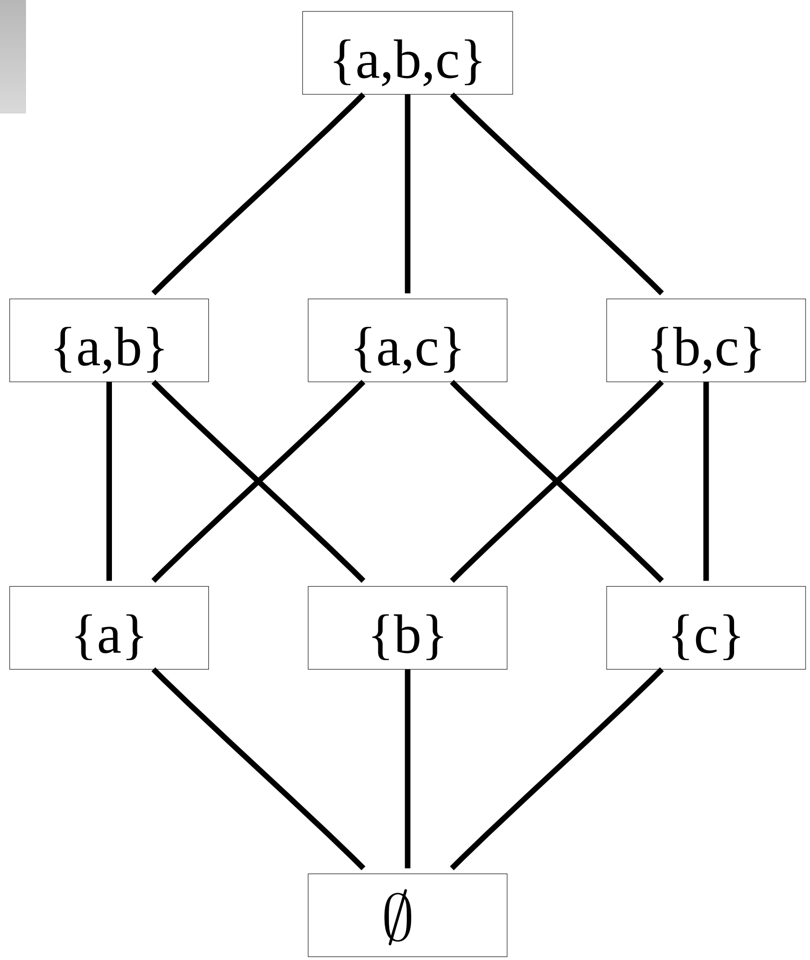


A very simple poset



The diagram of a poset A $= \langle\{\bot, a,\ b,\ \top\}, \leq\rangle$

A poset of sets

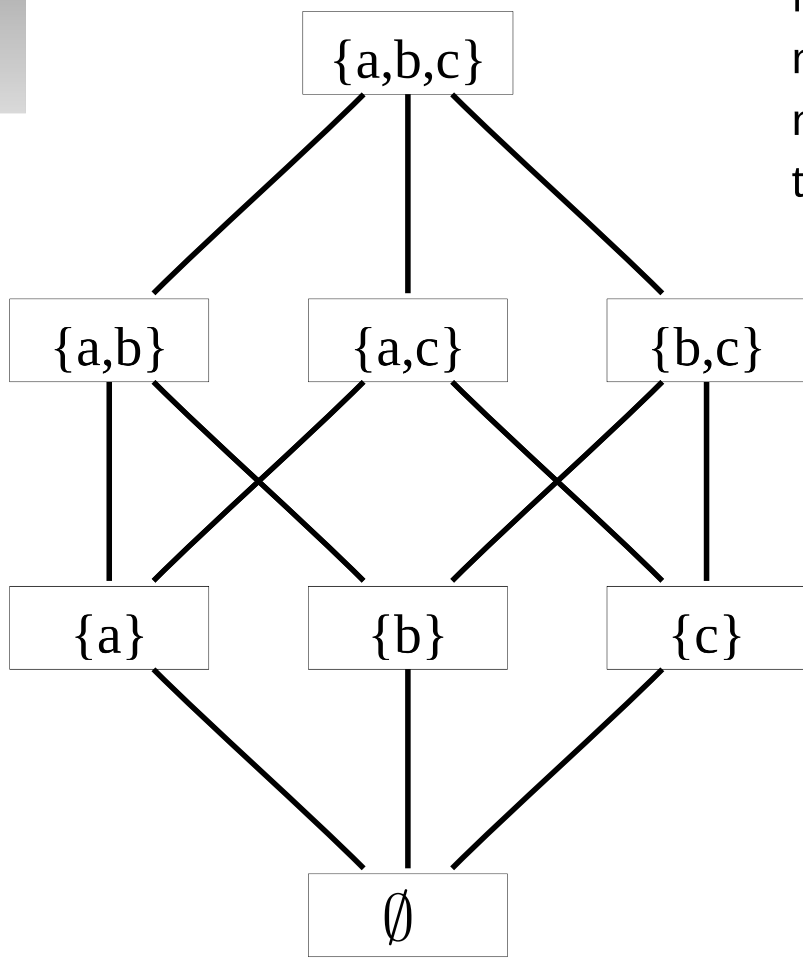


The diagram of the poset $\wp(\mathsf{A})$ for

$$\mathsf{A} = \{a,\ b,\ c\}$$

A line connecting a lower node to an upper node means the lower node is $\subseteq$ the upper.

Partial means partial

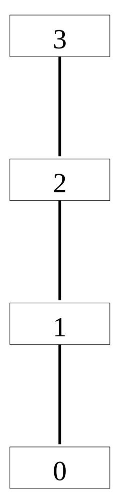


Note that not all sets are ordered by $\subseteq$. Thus no line connects $\{a,\ b\}$ and $\{b,\ c\}$ because neither is a subset of the other. We say these two sets are **incomparable** in that ordering.

Linear Orders

If an ordering has no incomparable elements, then it is **linear**. For example, the $\leq$ relation on numbers gives a **linear** ordering.

Definition 2. *A weak order is linear iff for every pair of elements a and b either $a \leq b$ or $b \leq a$:*

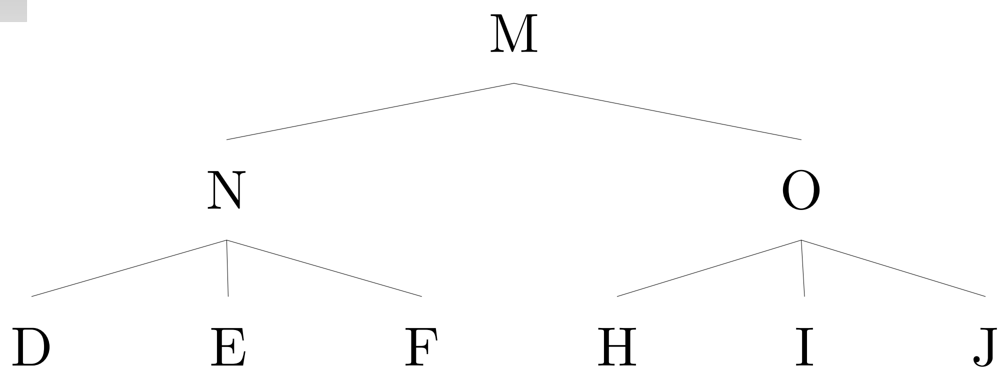


$\mathbb{N}$ under $<$: A poset with a linear order

$$\forall a, b[a \leq b \vee b \leq a]$$

Higgenbotham Dominance

$x \leq y$ means x dominates y

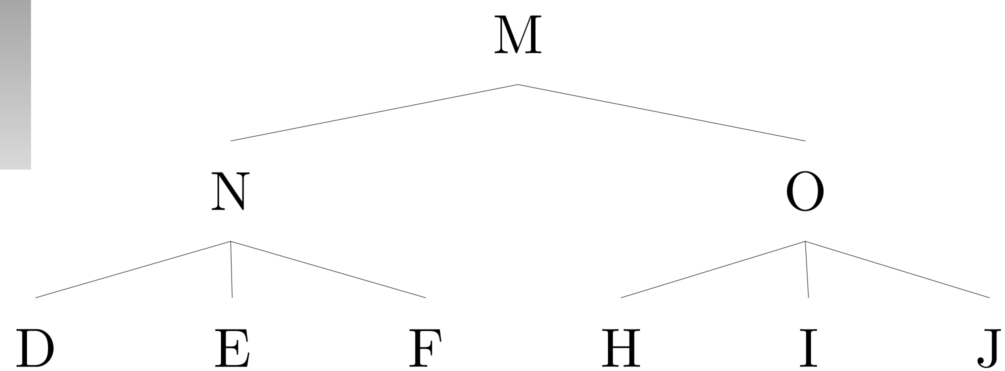


$\mathrm{M} \leq \mathrm{N}$	$\mathrm{M} \leq 0$	
$\mathrm{N} \leq \mathrm{D}$	$\mathrm{N} \leq \mathrm{E}$	$\mathrm{N} \leq \mathrm{F}$
$\mathrm{O} \leq \mathrm{H}$	$\mathrm{O} \leq \mathrm{I}$	$\mathrm{O} \leq \mathrm{J}$
$\mathrm{M} \leq \mathrm{D}$	$\mathrm{M} \leq \mathrm{E}$	$\mathrm{M} \leq \mathrm{F}$
$\mathrm{M} \leq \mathrm{H}$	$\mathrm{M} \leq \mathrm{I}$	$\mathrm{M} \leq \mathrm{J}$
$\mathrm{N} \not\leq \mathrm{H}$	$\mathrm{N} \not\leq \mathrm{I}$	$\mathrm{N} \not\leq \mathrm{J}$
$\mathrm{O} \not\leq \mathrm{D}$	$\mathrm{O} \not\leq \mathrm{E}$	$\mathrm{O} \not\leq \mathrm{F}$

Higgenbotham

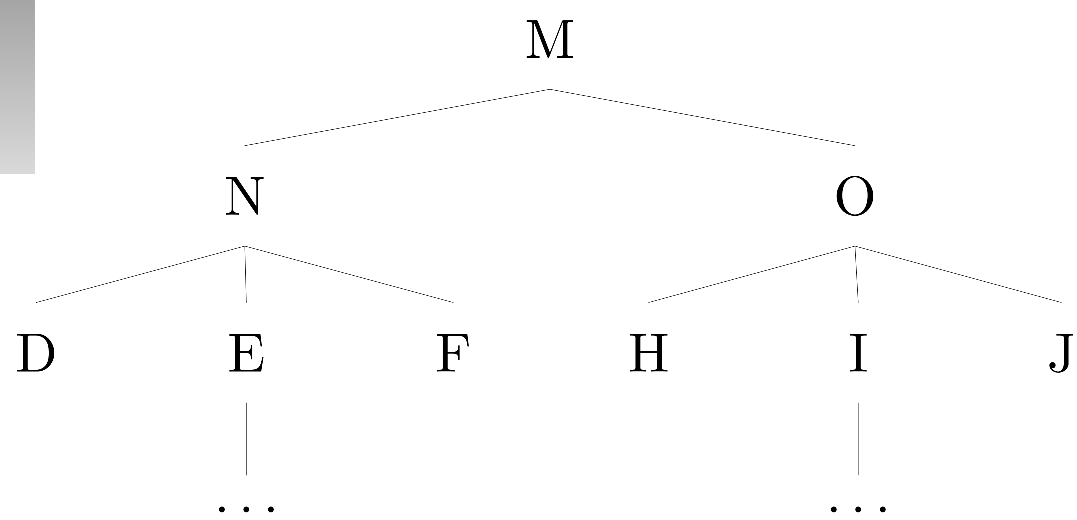
1. Reflexivity: $x \leq x$
2. Transitivity: If $x \leq y$ and $y \leq z$ then $x \leq z$
3. Antisymmetry: If $x \leq y \leq x$ then $x = y$
4. Single Mother: If $x \leq z$ and $y \leq z$ then either $x \leq y$ or $y \leq x$ (or both if $x = y$).

Reflexivity



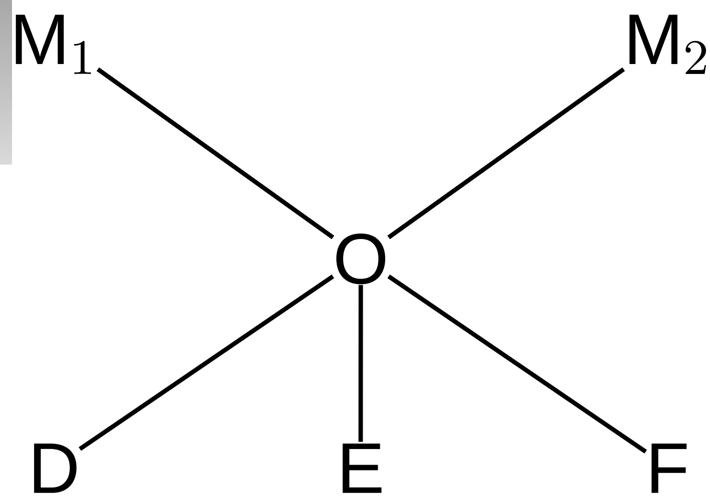
$M \leq M$	$N \leq N$	$O \leq O$
$D \leq D$	$E \leq E$	$F \leq F$
$H \leq H$	$I \leq I$	$J \leq J$

Transitivity



$\boxed{M \leq N}$	$M \leq 0$	
$N \leq D$	$\boxed{N \leq E}$	$N \leq F$
$O \leq H$	$O \leq I$	$O \leq J$
$E \leq \ldots$		$I \leq \ldots$
$M \leq D$	$\boxed{M \leq E}$	$M \leq F$
$M \leq H$	$M \leq I$	$M \leq J$
$M \leq \ldots$	$\ldots$	

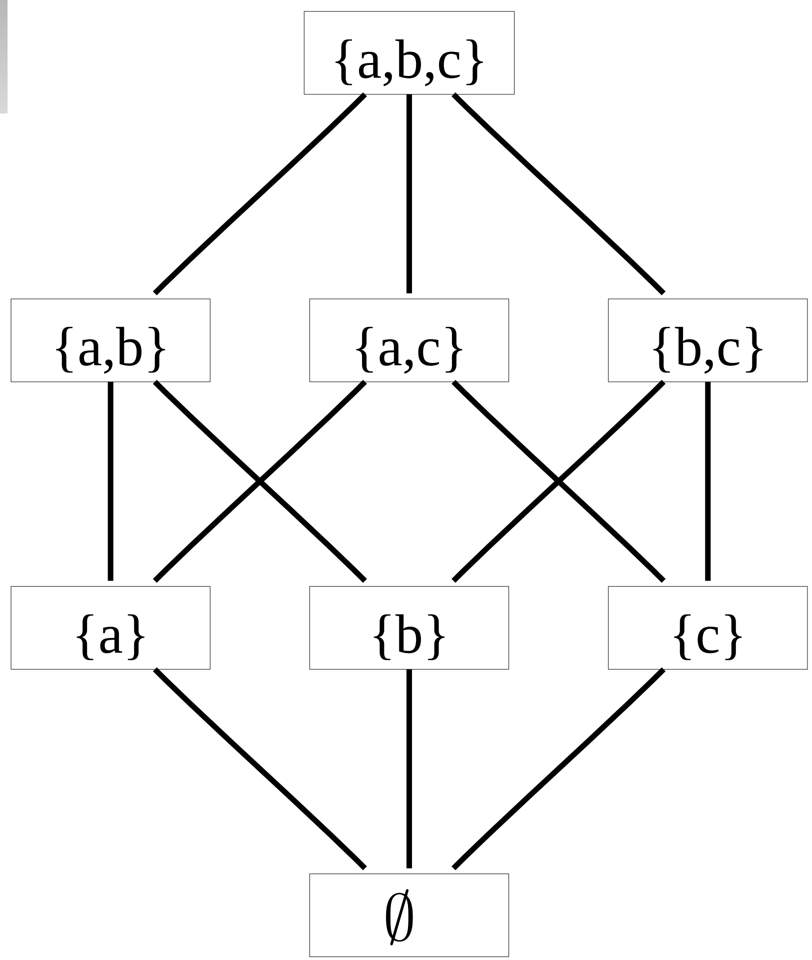
Single Mother



$$\frac{M_1 \leq O \qquad M_2 \leq O}{M_1 \leq M_2 \quad \text{No!} \qquad M_2 \leq M_1 \quad \text{No!}}$$

The single mother condition fails!

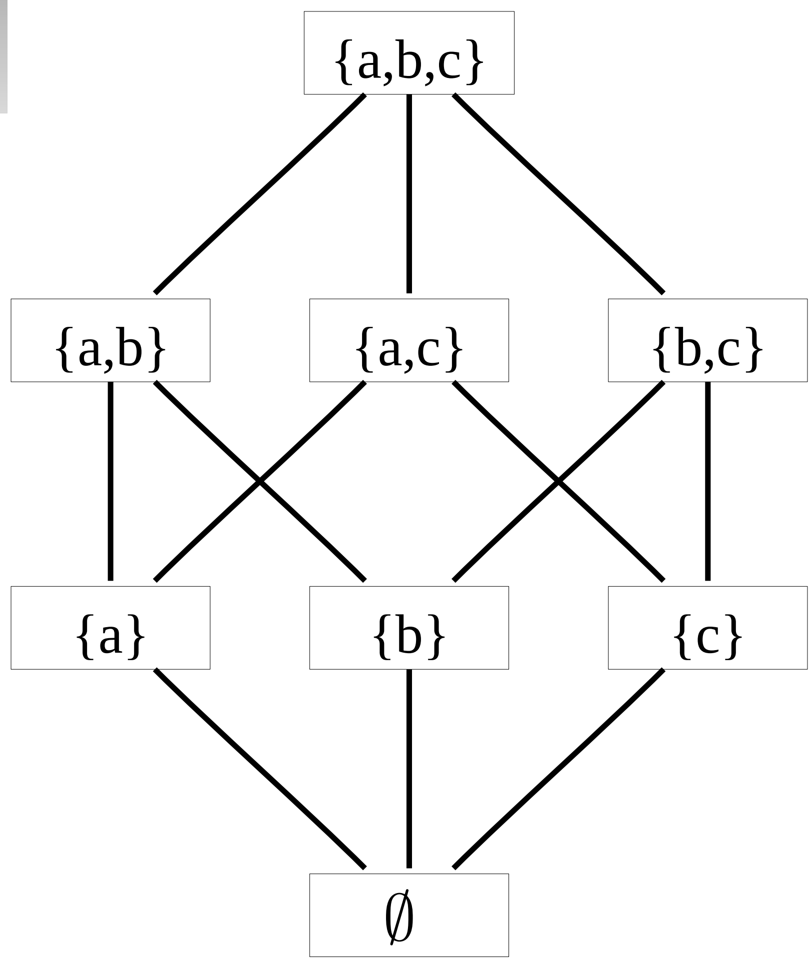
Lower bounds



$\{a,\ b\} \leq \{a,\ b,\ c\}$

We say $\{a,\ b\}$ is a **lower bound** for $\{a,\ b,\ c\}$.

Lower bounds for sets of things



$\{a,\ b\} \leq \{a,\ b,\ c\}$

$\{a,\ b\} \leq \{a,\ b\}$

Note that two sets may share a common lower bound. We generalize the notion lower bound to sets of elements of $\wp(\mathsf{A})$.

Definition of lower bound for a set

Definition 3. *An element of* $\wp(A)$, x, *is a lower bound of a subset of* $\wp(A)$, *S, if and only if, for every* $y \in S$,

$$x \subseteq y$$

Example

$\{b\}$ is a lower bound for $\{\{a,\ b\},\ \{b,\ c\}\}$ **because**

1. $\{b\} \subseteq \{a,\ b\}$; and
2. $\{b\} \subseteq \{b,\ c\}$.

Multiple lower bounds

A set of elements of $\wp(\text{A})$ may have multiple lower bounds.

1. The lower bounds of

$$\text{S} = \{\{a,\ b,\ c\},\ \{b,\ c\}\}$$

   are $\{b\}$, $\{c\}$, $\{b,\ c\}$ and $\emptyset$. There are no others.

2. Of the lower bounds of S, $\{b,\ c\}$ is the **greatest lower bound**.

Greatest Lower Bound and Least Upper Bound

Greatest Lower Bound

Definition 4. *An element a is the* **greatest lower bound of a set S** *(glb of S) if and only if:*

1. *a is a lower bound of S*
2. *For every lower bound b of S, $b \leq a$.*

In this case we write:

$$a = \textit{glb } S$$

Example: greatest lower bounds

1. The lower bounds of

$$\mathsf{S} = \{\{a,\ b,\ c\},\ \{b,\ c\}\}$$

   are $\{b\}$, $\{b,\ c\}$ and $\emptyset$.

2. Of the lower bounds of S, $\{b,\ c\}$ is the **greatest lower bound**.

3. In general, when A, B are sets,

$$\text{glb}\ \{\mathsf{A},\ \mathsf{B}\} = \mathsf{A} \cap \mathsf{B}$$

Least Upper Bound

Definition 5. *An element a is the* **least upper bound of a set S** *(lub of S) if and only if:*

1. *a is an upper bound of S*
2. *For every upper bound bound b of S, $a \leq b$.*

In this case we write:

$$a = \textit{lub } S$$

Example: least upper bounds

1. Within the poset $\wp\{a, b, c\}$, the upper bounds of

$$\mathsf{S} = \{\{a\}, \{b\}\}$$

   are $\{a, b\}$ and $\{a, b, c\}$.

2. Of the upper bounds of S, $\{a, b\}$ is the **least upper bound**.

3. In general, when A, B are sets,

$$\text{lub } = \{\mathsf{A}, \mathsf{B}\} = \mathsf{A} \cup \mathsf{B}$$

Meet: A greatest lower bound operation

Consider two elements p and q of a poset. We define

$$p \wedge q \stackrel{\Delta}{=} \text{glb}\ \{p,\ q\}$$

$p \wedge q$ is read "p **meet** q".

Algebra?

$\wedge$ is an operation. Can we use that operation to define an algebra?

1. In an **algebra**, a two-place operation needs to be defined for every pair of elements in the algebra.
2. In an arbitrary poset the greatest lower bound of a set of elements is not guaranteed to exist.

Join: A least upper bound operation

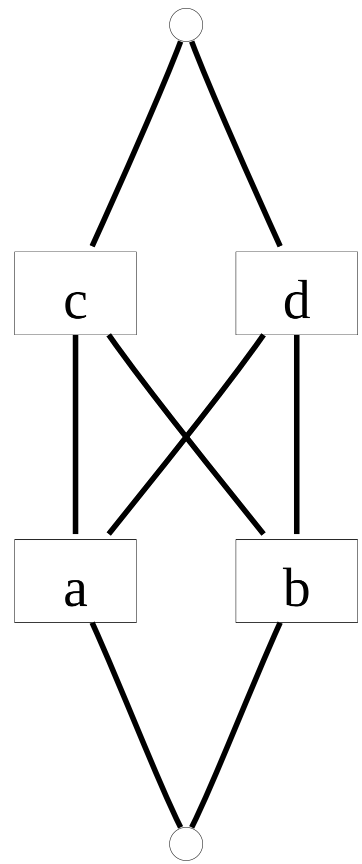
Consider two elements p and q of a poset. We define

$$p \vee q \triangleq \text{lub}\ \{p,\ q\}$$

$p \vee q$ is read "p **join** q".

In an arbitrary poset the least upper bound of a set is not guaranteed to exist.

LUB and GLB do not always exist



A poset in which least upper and greatest lower bounds do not always exist.

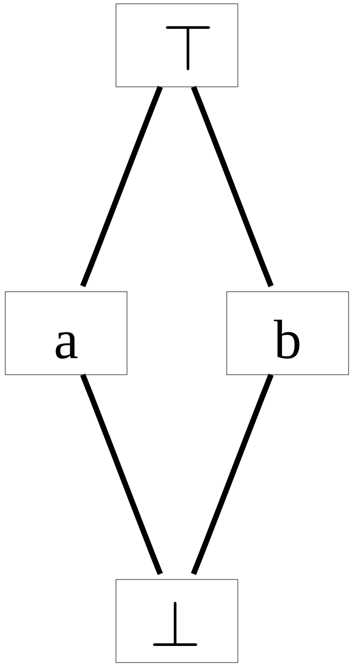
Introducing lattice posets

Definition 6. *A lattice is a poset A in which, for every* $p, q \in A$*,*

$$p \wedge q \in A$$
$$p \vee q \in A$$

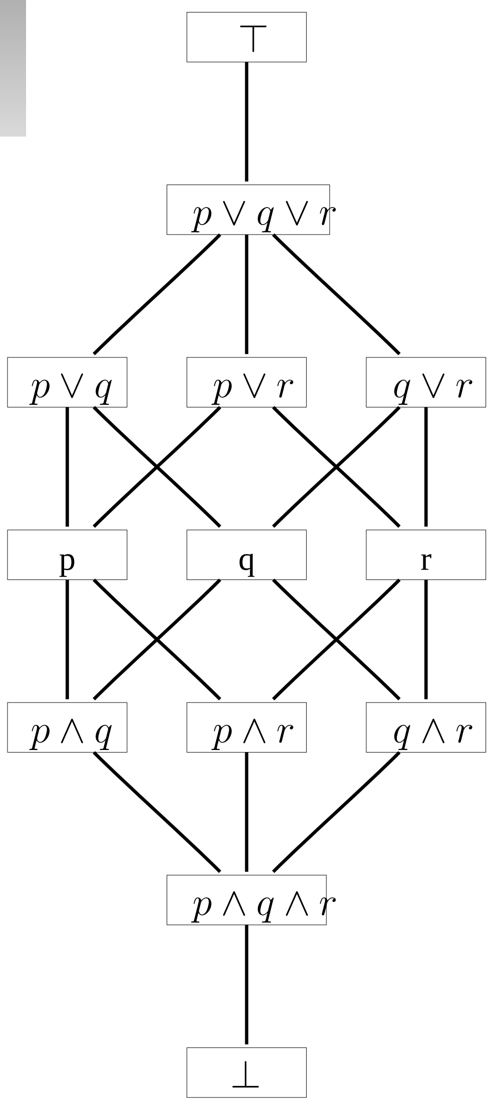
A lattice poset is an **algebra** $\langle A, \wedge, \vee \rangle$ because $\wedge$ and $\vee$ are operations satisfying the closure and uniqueness requirements (by the above definition).

A very simple lattice poset



The diagram of a simple lattice poset A $= \langle\{\bot, a,\ b,\ \top\}, \leq\rangle$

A less simple lattice

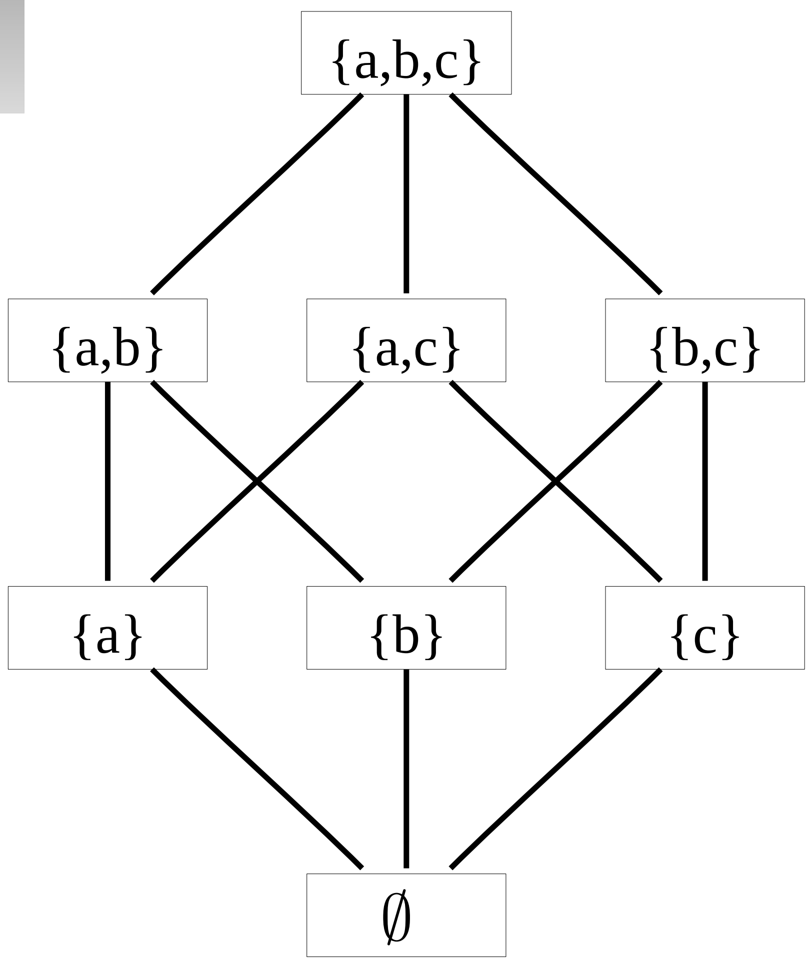


Summary: posets and bounds

1. A **poset** is a set A together with a weak order, written $\leq$. We write this $\langle$A, $\leq\rangle$, or sometimes just A.
2. For any element $x, y \in$ A, we say x is a **lower/upper bound** of y iff $x \leq y / y \leq x$.
3. For any subset S of A, we say x is **a lower/upper bound of the set S** iff for every $y \in$ S, $x \leq y / y \leq x$.
4. For any subset S of A, we say x is **a greatest lower/least upper bound of the set S** iff for every lower/upper bound y of S, $y \leq x / x \leq y$. We write

$$x = \text{glb S} \quad \text{greatest lower bound}$$
$$x = \text{lub S} \quad \text{least upper bound}$$

The lattice poset of sets



The poset $\wp(\mathsf{A})$ for

$$\mathsf{A} = \{a,\ b,\ c\}$$

turns out to be a lattice poset.

Summary: Lattice posets

1. A lattice poset is a poset A in which for any two elements a and b,

$$\text{glb}\ \{a,\ b\} \in \text{A}$$
$$\text{lub}\ \{a,\ b\} \in \text{A}$$

2. We define two operations $\wedge$ and $\vee$:

$$a \wedge b = \text{glb}\ \{a,\ b\}$$
$$a \vee b = \text{lub}\ \{a,\ b\}$$

3. $\langle \text{A},\ \wedge,\ \vee \rangle$ is an algebra. By definition, $\wedge$ and $\vee$ satisfy closure and uniqueness.
4. In-class exercise: Prove uniqueness for $\wedge$ and $\vee$.