Relations

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A New Universe of Discourse

Example 1. *English Obstruents*

Manner	**Location**	*Lab.*	*LabVel.*	*Dent.*	*Alv.*	*Alv.-Pal.*	*Vel.*	*Glot.*
Stops	**Oral**	*p/b*			*t/d*		*k/g*	
	Nasal	*m*			*n*		*ŋ*	
Fricatives	**Oral**		*f/v*	*θ/ð*	*s/z*	*ʃ/ʒ*		*h*

StopFric | SameVoice | SamePlace |

New Sets

Obstruents $=$ { p, b, t, d, k, g, m, n, ŋ, s, z, θ, ð, ʃ, ʒ, h}	$\vert$Obstruents$\vert = 16$
Stop $=$ {p, b, t, d, k, g, m, n, ŋ}	$\vert$Stops$\vert = 9$
Fric $=$ {θ, ð, s, z, ʃ, ʒ, h}	$\vert$Fricatives$\vert = 7$
Nas $=$ {m, n, ŋ}	$\vert$Nasals$\vert = 3$
Oral $=$ {p, b, t, d, s, z, k, g, θ, ð, ʃ, ʒ, h}	$\vert$Orals$\vert = 13$
OralStop $=$ {p, b, t, d, k, g}	$\vert$OralStops$\vert = 6$

Introducing Ordered Pairs

- Ordered Pairs

$$\langle a,\, b\rangle$$

- Order matters

$$\langle a,\, b\rangle \neq \langle b,\, a\rangle$$

- Not like sets

$$\{\text{a, b}\} = \{\text{b, a}\}$$

Ordered pairs versus sets: More differences

- The same object can occur more than once

$$\langle a, a\rangle \neq \text{a} \neq \langle \text{a}\rangle \neq \{\text{a}\}$$

- Unlike sets

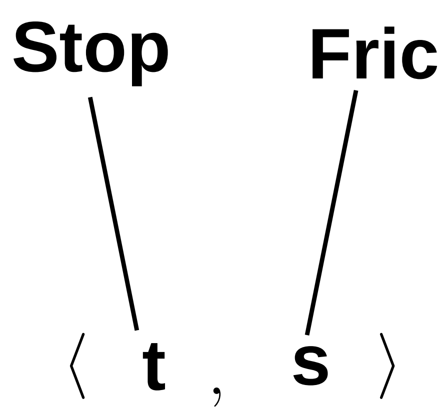
$$\{\text{a}\} = \{\text{a}, \text{a}\} = \{\text{a}, \text{a}, \text{a}\} \dots$$

Sets of Ordered pairs [Relations]

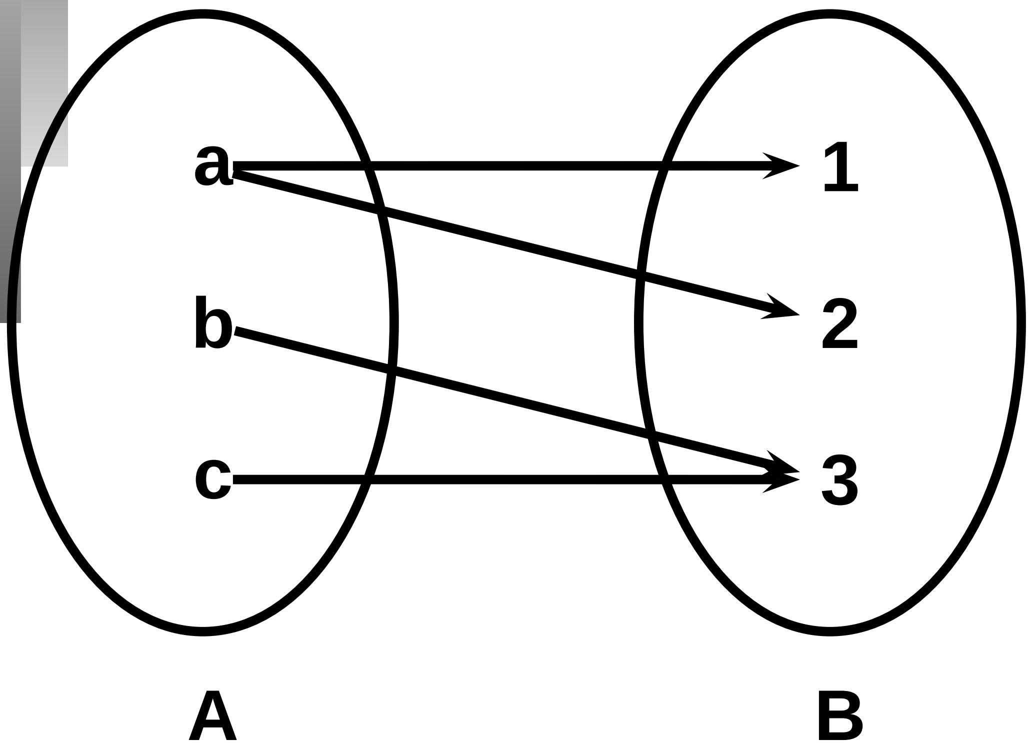
- A set of ordered pairs

$$\text{StopFric} = \{\ \langle t, s\rangle,\quad \langle t, z\rangle,\quad \langle d, s\rangle,\\ \langle d, z\rangle,\quad \langle n, s\rangle,\quad \langle n, z\rangle\}$$

- Each *first* member is from Stop, each *second* member is from Fric:



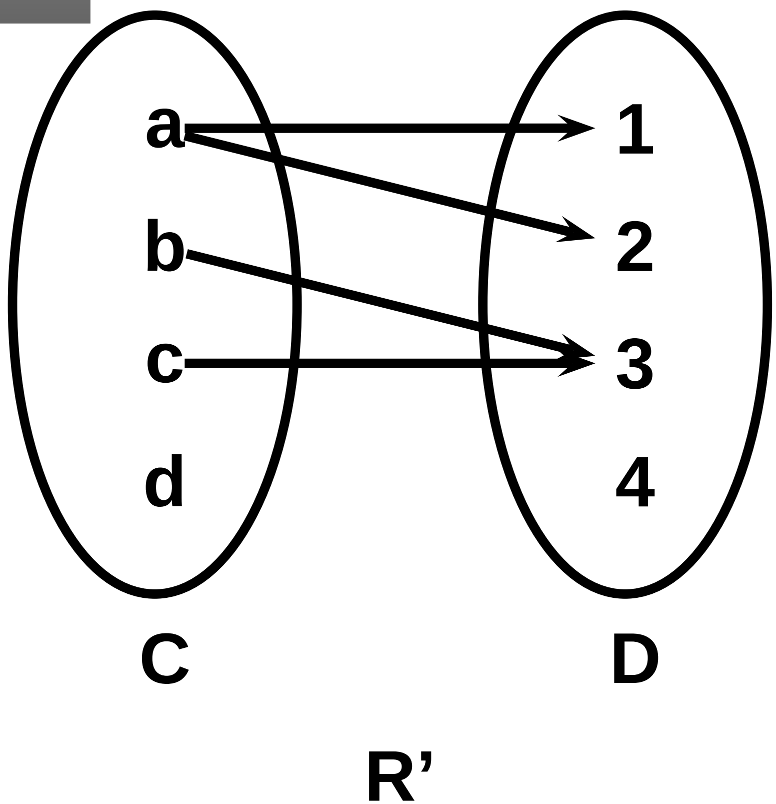
A Relation named R from A to B



$$\mathsf{R} = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 3\rangle, \langle c, 3\rangle\}$$

Domain and Range of a relation

A relation R' from C to D The subsets of C and D actually standing in R' are called the domain and range of R'.



$\text{Dom}(\text{R'}) = \{\text{a}, \text{b}, \text{c}\}$

$\text{Range}(\text{R'}) = \{1, 2, 3\}$

Cartesian Products

$\text{Stop} \times \text{Fric} = \{\langle x, y\rangle \mid \text{x} \in \text{Stop and y} \in \text{Fric}\}$

$$\{\ \langle p, \text{θ}\rangle, \langle p, \text{ð}\rangle, \langle p, s\rangle, \langle p, z\rangle, \langle p, \text{ʃ}\rangle, \langle p, \text{ʒ}\rangle, \langle p, x\rangle,$$
$$\langle t, \text{θ}\rangle, \langle t, \text{ð}\rangle, \langle t, s\rangle, \langle t, z\rangle, \langle t, \text{ʃ}\rangle, \langle t, \text{ʒ}\rangle, \langle t, x\rangle,$$
$$\langle k, \text{θ}\rangle, \langle k, \text{ð}\rangle, \langle k, s\rangle, \langle k, z\rangle, \langle k, \text{ʃ}\rangle, \langle k, \text{ʒ}\rangle, \langle k, x\rangle,$$
$$\langle b, \text{θ}\rangle, \langle b, \text{ð}\rangle, \langle b, s\rangle, \langle b, z\rangle, \langle b, \text{ʃ}\rangle, \langle b, \text{ʒ}\rangle, \langle b, x\rangle,$$
$$\langle d, \text{θ}\rangle, \langle d, \text{ð}\rangle, \langle d, s\rangle, \langle d, z\rangle, \langle d, \text{ʃ}\rangle, \langle d, \text{ʒ}\rangle, \langle d, x\rangle,$$
$$\langle g, \text{θ}\rangle, \langle g, \text{ð}\rangle, \langle g, s\rangle, \langle g, z\rangle, \langle g, \text{ʃ}\rangle, \langle g, \text{ʒ}\rangle, \langle g, x\rangle,$$
$$\langle m, \text{θ}\rangle, \langle m, \text{ð}\rangle, \langle m, s\rangle, \langle m, z\rangle, \langle m, \text{ʃ}\rangle, \langle m, \text{ʒ}\rangle, \langle m, x\rangle,$$
$$\langle n, \text{θ}\rangle, \langle n, \text{ð}\rangle, \langle n, s\rangle, \langle n, z\rangle, \langle n, \text{ʃ}\rangle, \langle n, \text{ʒ}\rangle, \langle n, x\rangle,$$
$$\langle \text{ŋ}, \text{θ}\rangle, \langle \text{ŋ}, \text{ð}\rangle, \langle \text{ŋ}, s\rangle, \langle \text{ŋ}, z\rangle, \langle \text{ŋ}, \text{ʃ}\rangle, \langle \text{ŋ}, \text{ʒ}\rangle, \langle \text{ŋ}, x\rangle\}$$

Order in Cartesian Products

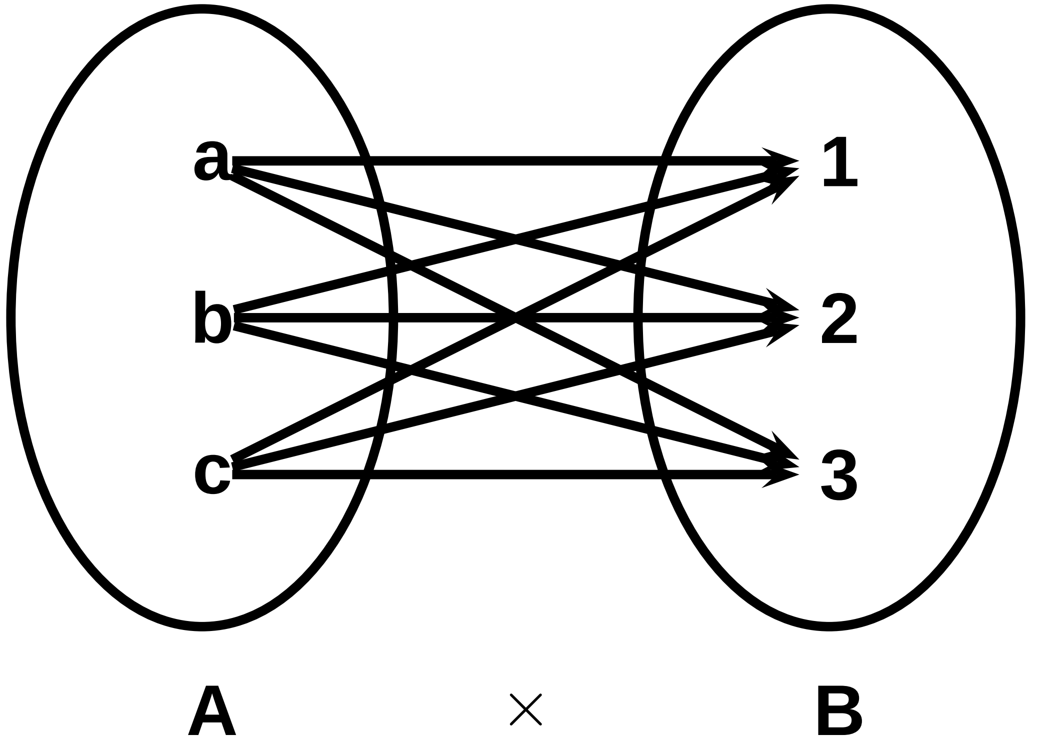
- Stop × Fric is order dependent, because the ordered-pairs in the set are:

$$\text{Fric} \times \text{Stop} \neq \text{Stop} \times \text{Fric}$$
$$\langle s,\, t \rangle \in \text{Fric} \times \text{Stop}$$
$$\langle s,\, t \rangle \notin \text{Stop} \times \text{Fric}$$
$$\langle t,\, s \rangle \notin \text{Fric} \times \text{Stop}$$
$$\langle t,\, s \rangle \in \text{Stop} \times \text{Fric}$$

- Stop × Fric is a special case of a relation from Stop to Fric. For any such relation R,

$$\text{R} \subseteq \text{Stop} \times \text{Fric}$$

The Cartesian product of A and B



$$
\begin{aligned}
\mathsf{A} \times \mathsf{B} \;=\; \{\; & \langle a,\,1\rangle, \quad \langle a,\,2\rangle, \quad \langle a,\,3\rangle, \\
& \langle b,\,1\rangle, \quad \langle b,\,2\rangle, \quad \langle b,\,3\rangle, \\
& \langle c,\,1\rangle, \quad \langle c,\,2\rangle, \quad \langle c,\,3\rangle \;\}
\end{aligned}
$$

Cardinality of Cartesian Product

Observation: To construct $A \times B$ we need to pair each member of A with all $|B|$ members of B. Since we have to do that $|A|$ times

$$|A \times B| = |A| \times |B|$$

Identity Relation

A special relation from any set A to itself is the identity relation:

$$\mathsf{I}_{\mathrm{A}} = \{\langle x, x\rangle \mid x \in \mathsf{A}\}$$

SamePlace, SameVoicing

Relations can be defined by listing OR by predicates, just as sets can be. Reassuring because they are sets.

Example 2. *SamePlace*

$$SamePlace = \{\langle x, y\rangle \mid \ \langle x, y\rangle \in Obstruents \times Obstruents \text{ and } x \text{ and } y \text{ are articulated in the same plac}$$

Example 3. *SameVoicing*

$$SameVoicing = \{ \ \langle x, y\rangle \mid \langle x, y\rangle \in Obstruents \times Obstruents \text{ and } x \text{ and } y \text{ have the same voicing.}\}$$

These are relations from Obstruents to Obstruents.

Example 4. *StopFric*

$$\textit{StopFric} = \{\langle x, y\rangle \mid \ \langle x, y\rangle \in \textit{Stop} \times \textit{Fric and } \textit{SamePlace}(x, y)\}$$

Consulting example 1, we find :

$$\textit{StopFric} = \{\ \langle t, s\rangle, \quad \langle t, z\rangle, \quad \langle d, s\rangle, \quad \langle d, z\rangle, \quad \langle n, s\rangle, \quad \langle n, z\rangle\}$$

Definition 1. *Inverse of a Relation*
We define the inverse $\mathbf{R}^{-1}$ *or a relation* $\mathbf{R}$ *as follows:*

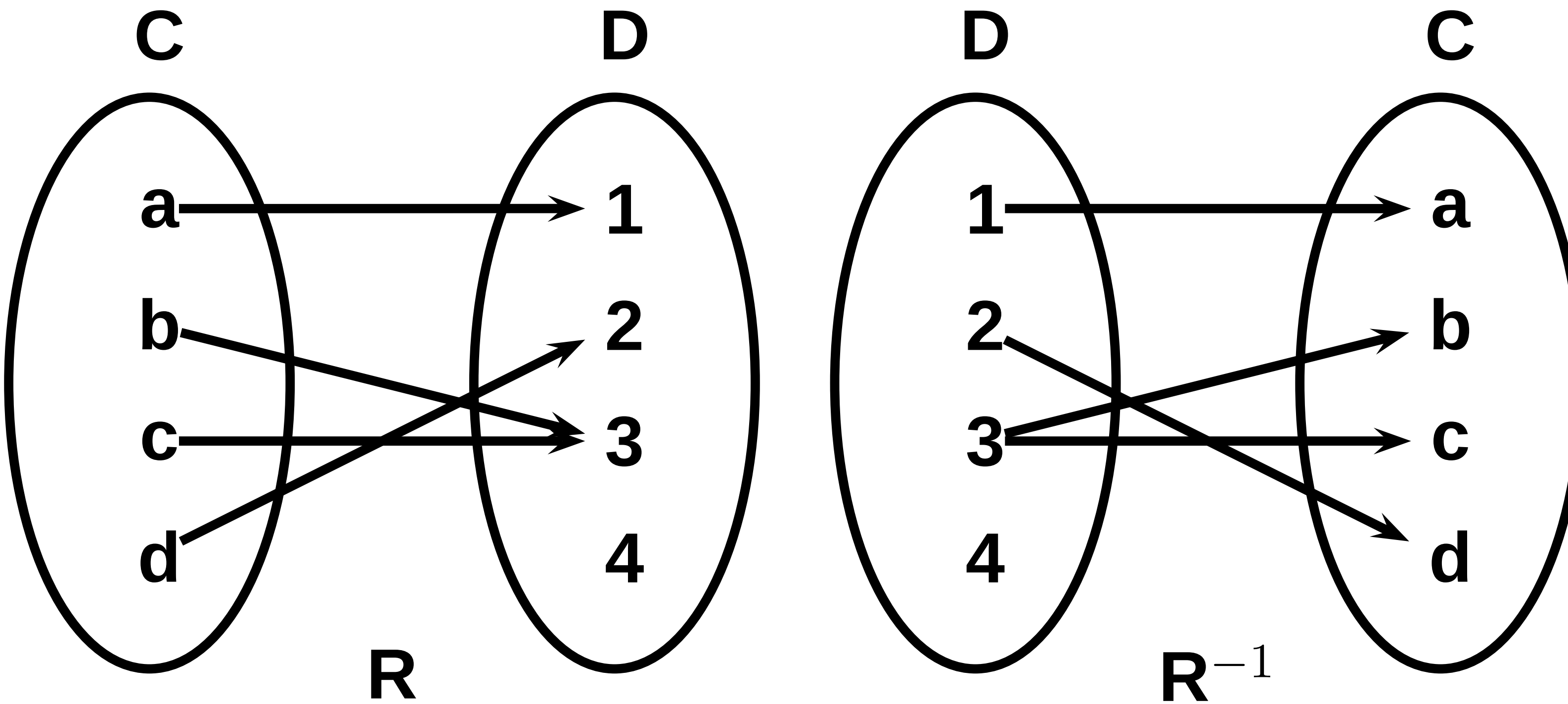
$$\mathbf{R}^{-1} = \{\langle x,\ y\rangle \mid \langle y,\ x\rangle \in R\,\}$$

Note that it is frequently the case that:

$$\mathbf{R}^{-1} \bigcap \mathbf{R} = \emptyset$$

When is it not?

A relation and its inverse



Inverse Example

Example 5. *Inverse of StopFric*

$$\begin{aligned} \mathit{StopFric}^{-1} &= \{\langle s, t\rangle, \langle z, t\rangle, \langle s, d\rangle, \langle z, d\rangle, \langle s, n\rangle, \langle z, n\rangle\} \\ &= \mathit{FricStop} \end{aligned}$$

Relation Composition

Definition 2.

$$R_1 \circ R_2 = \{\langle x,\, z\rangle \mid \exists y[\langle x,\, y\rangle \in R_2 \textit{ and } \langle y,\, z\rangle \in R_1]\,\}$$

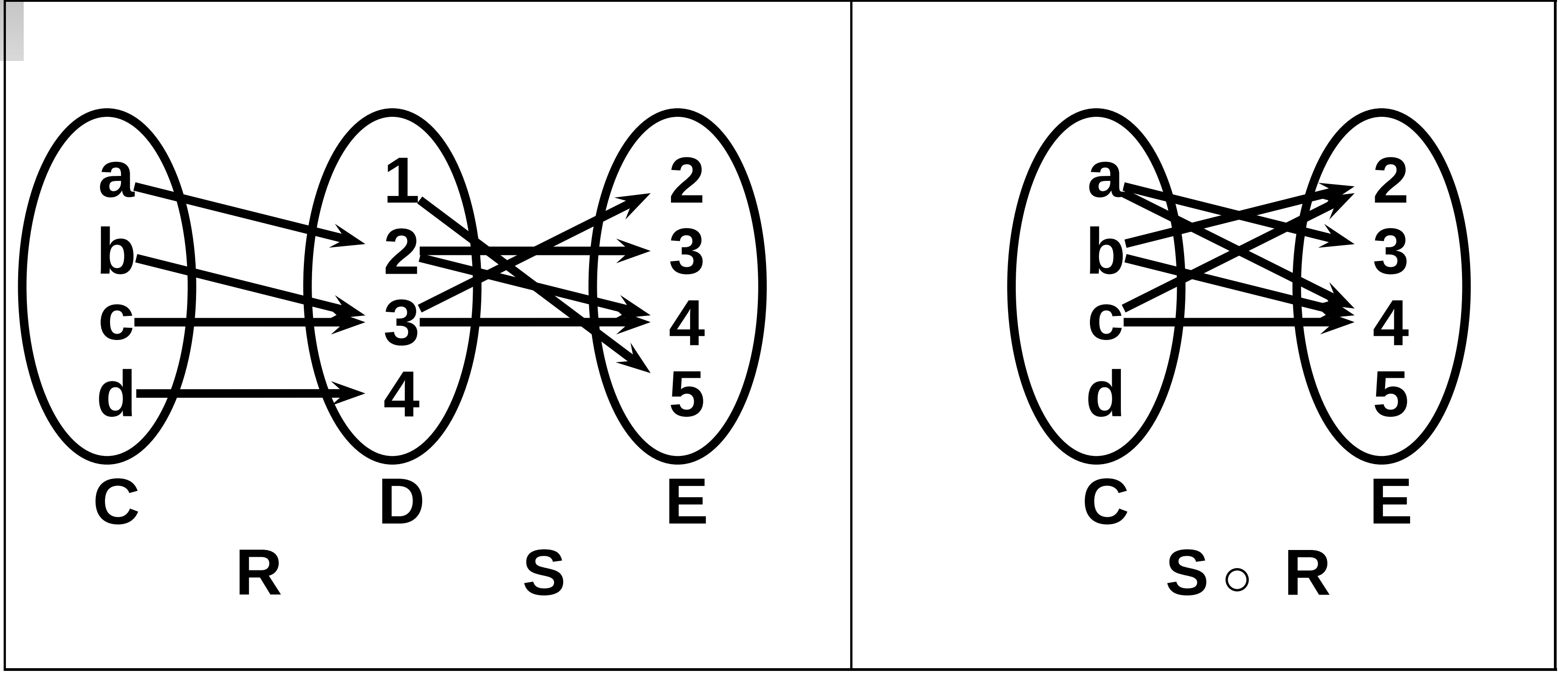
We introduce a new piece of notation here $\exists$, which means “there exists”.

$$\mathrm{R}_1 \circ \mathrm{R}_2 = \{\langle x,\, z\rangle \mid \text{ There exists a } y \text{ such that } \langle x,\, y\rangle \in \mathrm{R}_2 \text{ and } \langle y,\, z\rangle \in \mathrm{R}_1 \text{ . } \}$$

Note:

If R_2 is from A to B and R_1 is from C to D then $\mathrm{R}_1 \circ \mathrm{R}_2$ is from A to D.

Composition of relations R and S.



Although S○R must be from the same set as R and to the same set as S it does not have to have the same domain as R or the same range as S.

Defining SamePlace using relation composition

Example 6. *Places*

$$Places = \{bilabial,\ interdental,\ alveolar,\ palatal,\ velar\,\}$$

Example 7. *SoundPlace*

$$\begin{aligned} SoundPlace = \{\langle x, y\rangle \mid\ & x \in Obstruents \text{ and } y \in Places \text{ and} \\ & x \text{ is pronounced at place } y\,\} \end{aligned}$$

Example 8. *SamePlace defined by composition*

$$SamePlace = SoundPlace^{-1} \circ SoundPlace$$

Informal proof

Example 9. *Following the relation links for one pair*

$$\langle t,\ alveolar \rangle \in \mathit{SoundPlace}$$

Now $\mathit{SoundPlace}^{-1} \circ \mathit{SoundPlace}$ *includes all pairs* $\langle t,\ x \rangle$ *such that*

$$\langle alveolar,\ x \rangle \in \mathit{SoundPlace}^{-1}$$

That is:

$$\{\langle alveolar,\ t \rangle,\ \langle alveolar,\ d \rangle,\ \langle alveolar,\ s \rangle,\ \langle alveolar,\ z \rangle\}$$

Thus $\mathit{SoundPlace}^{-1} \circ \mathit{SoundPlace}$ *includes*

$$\{\langle t,\ t \rangle,\ \langle t,\ d \rangle,\ \langle t,\ s \rangle,\ \langle t,\ z \rangle\}$$

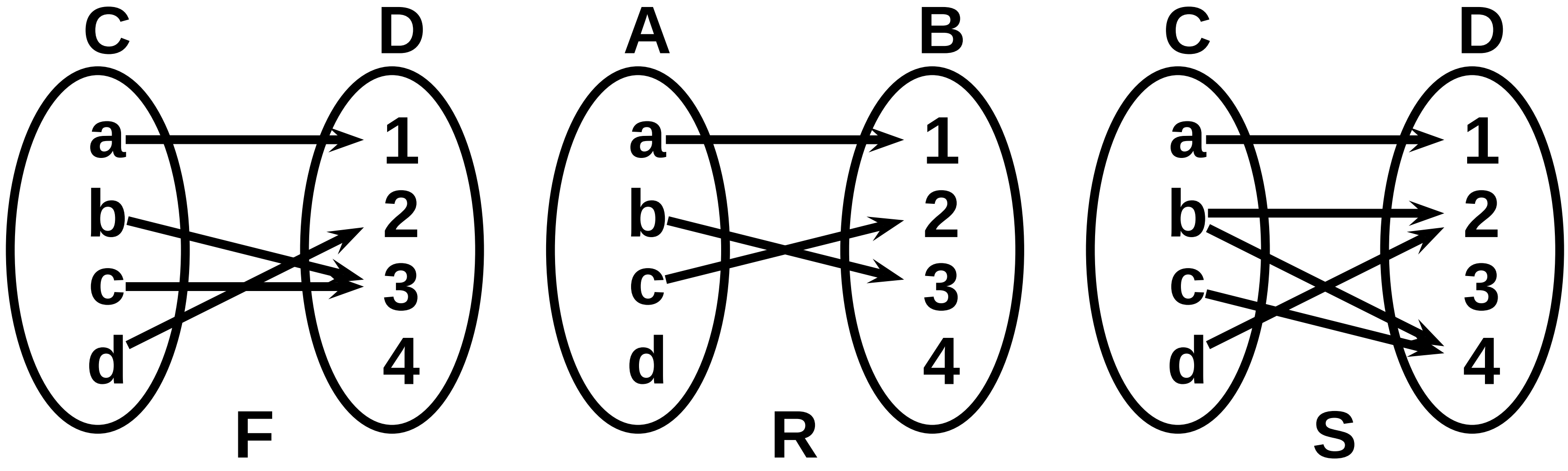
Functions

A special case of relations

Definition 3. *Functions*
A relation R from A to B is a function from A to B if and only if

1. *For each member x of the domain of R, there is a* unique *member of the range standing in relation R to x.*
2. *Dom(R) = A.*

Functions and non functions

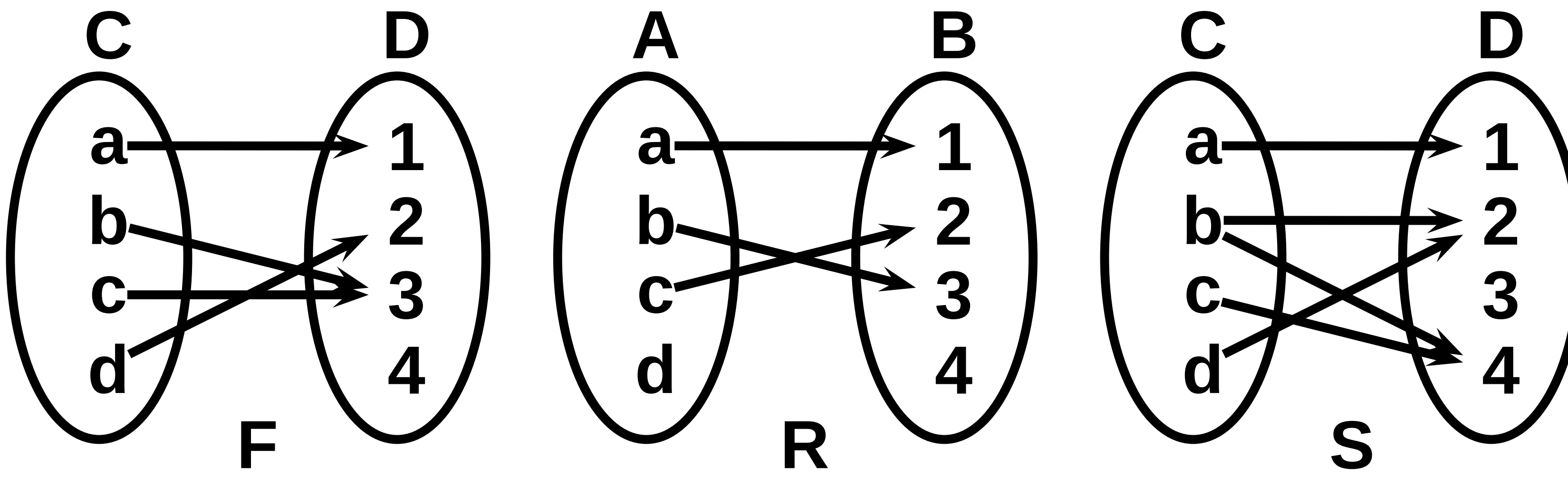


F

R

S

Functions and non functions

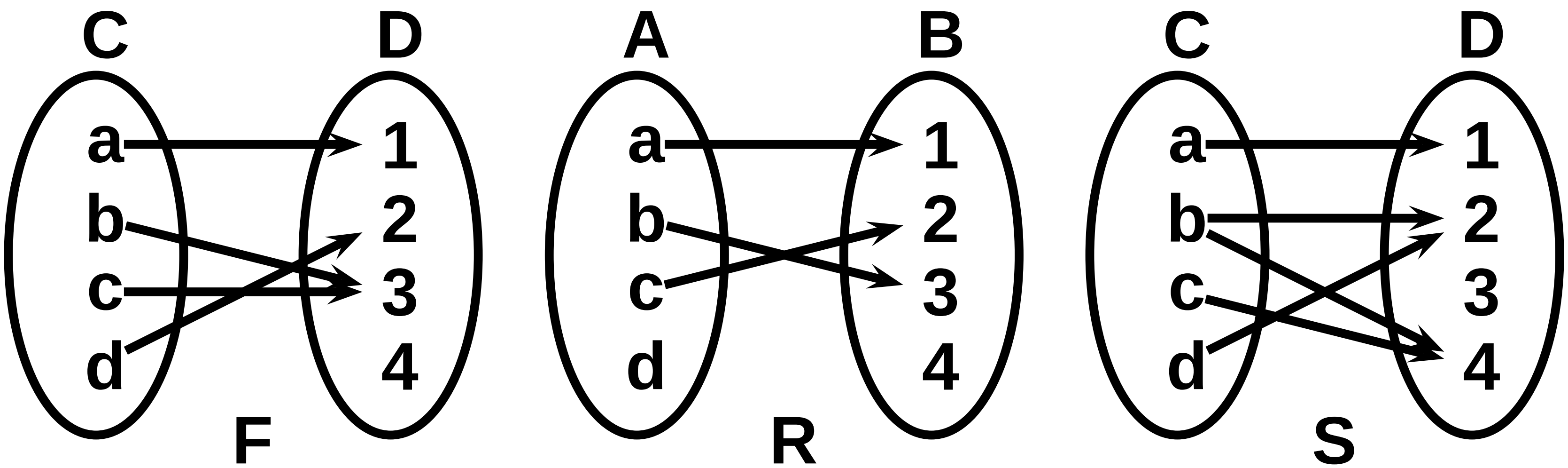


F is a function from C to D.

R

S

Functions and non functions

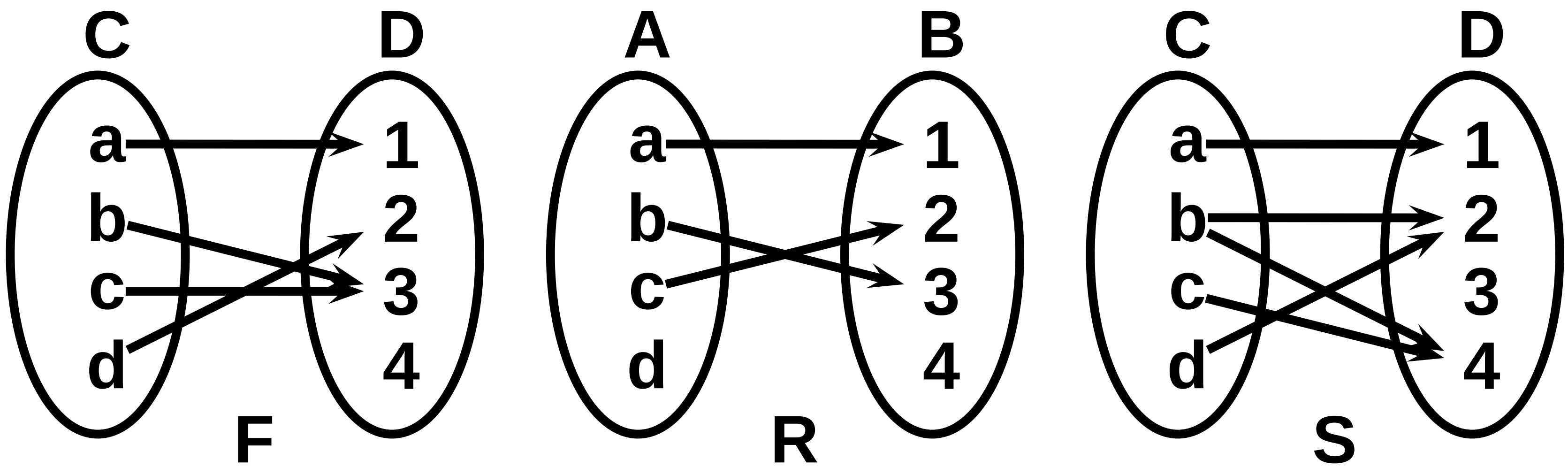


F is a function from C to D.

R is not a function from A to B (Clause 2)

S

Functions and non functions



F is a function from C to D.

R is not a function from A to B (Clause 2)

S is not a function (Clause 1)

Another function

- **SoundPlace** is a function from obstruents to places because each obstruent has exactly one place it is pronounced at.
- **SoundPlace**$^{-1}$ is NOT a function because each position may have many sounds pronounced at it. For example

$$\langle alveolar,\ s\rangle \in \text{SoundPlace}^{-1}$$
$$\langle alveolar,\ t\rangle \in \text{SoundPlace}^{-1}$$
$$\langle alveolar,\ d\rangle \in \text{SoundPlace}^{-1}$$
$$\langle alveolar,\ z\rangle \in \text{SoundPlace}^{-1}$$
$$\langle alveolar,\ n\rangle \in \text{SoundPlace}^{-1}$$

The successor relation

- The successor relation in arithmetic is a function from integers to integers:

$$
\begin{aligned}
s(0) &= 1 \quad (1)\\
s(1) &= 2 \quad (2)\\
s(2) &= 3 \quad (3)\\
&\vdots \quad (4)\\
s(n) &= n+1 \quad (5)
\end{aligned}
$$

- For each integer there is a unique next number. Every integer HAS a successor and every integer has exactly one.

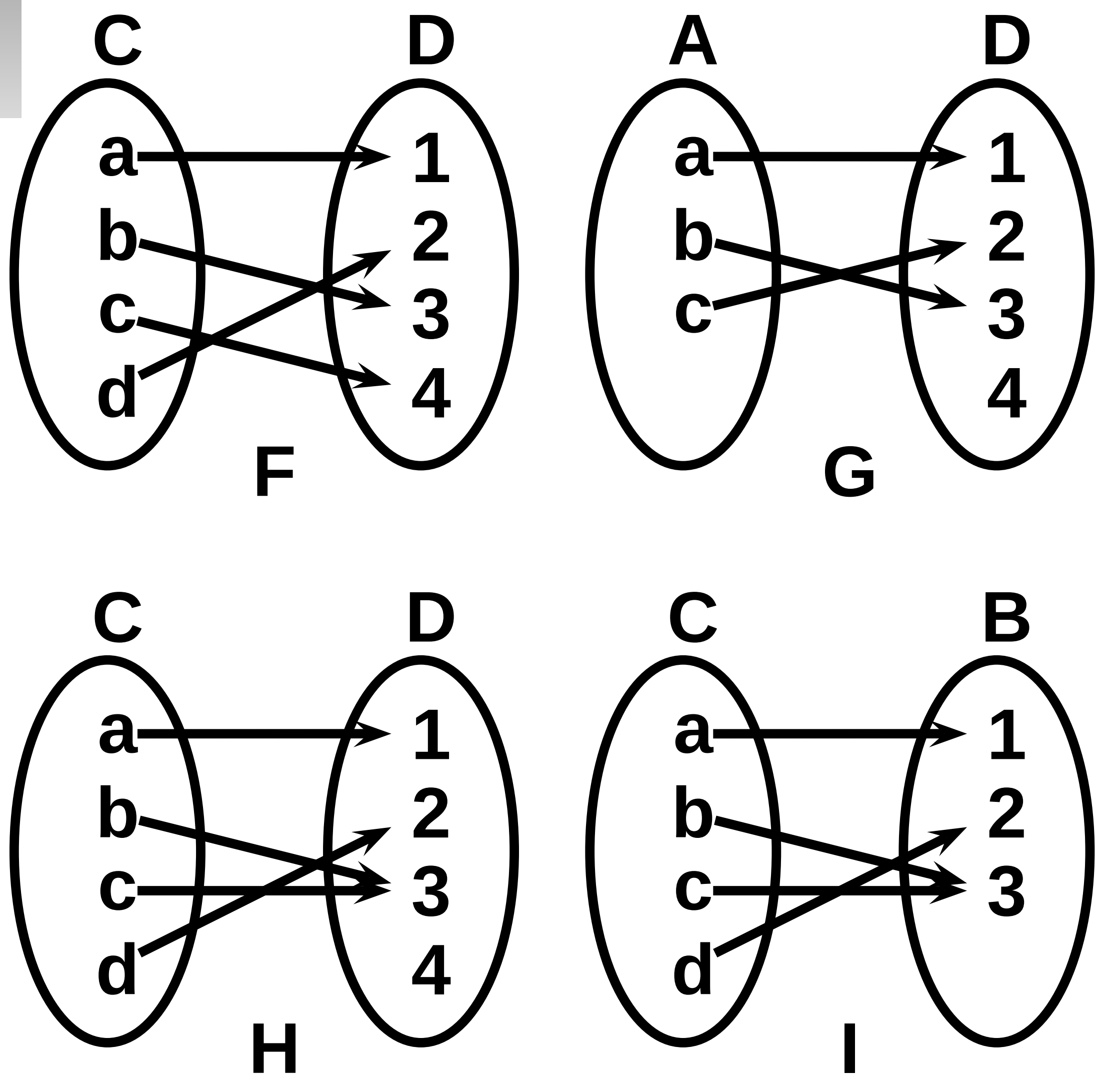
One-to-one functions

Definition 4. *We call a function F from A to B* ***one-to-one*** *(or injective) if and only if each member of F is paired with exactly one member of B.*

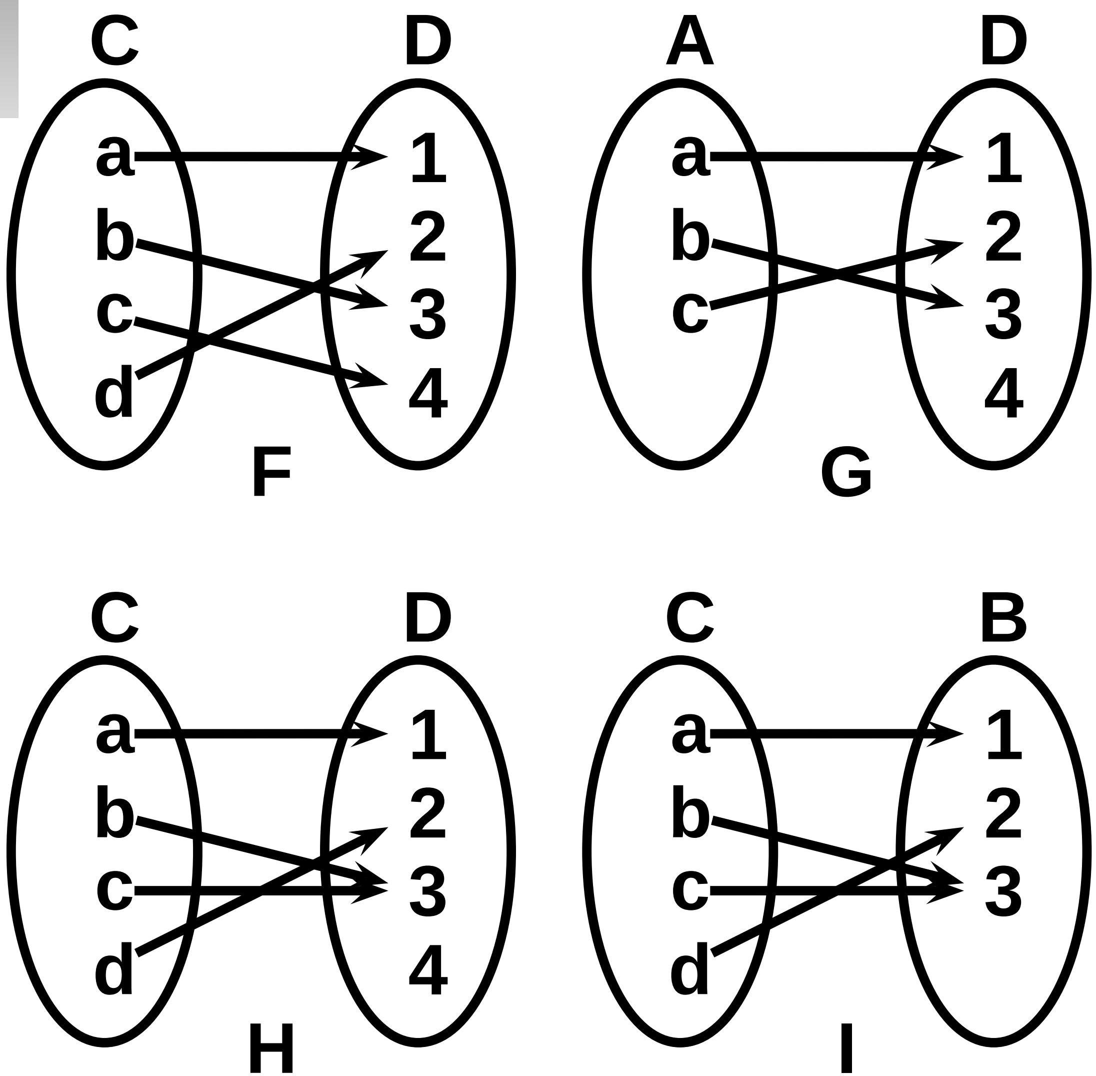
Onto functions

Definition 5. *We call a function* **onto** *(or surjective) if the range is equal to B. The term* into *does not mean the opposite of* one-to-one. *It's just a synonym for* to. *A function from A to B is also called a function from A* into *B, with no implied claim about whether it is onto or not.*

One-to-one and onto functions

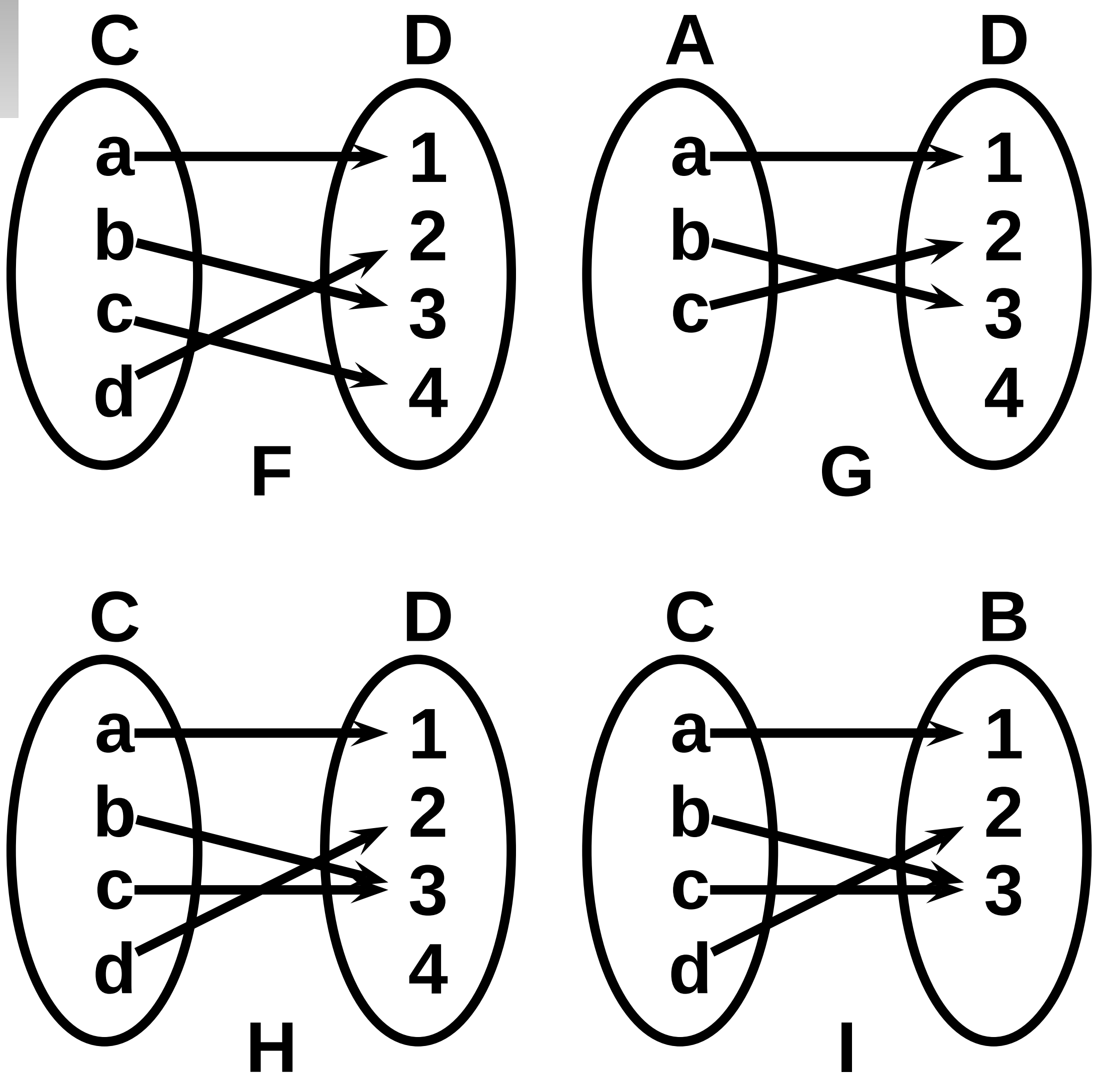


One-to-one and onto functions



F: 1-1, onto

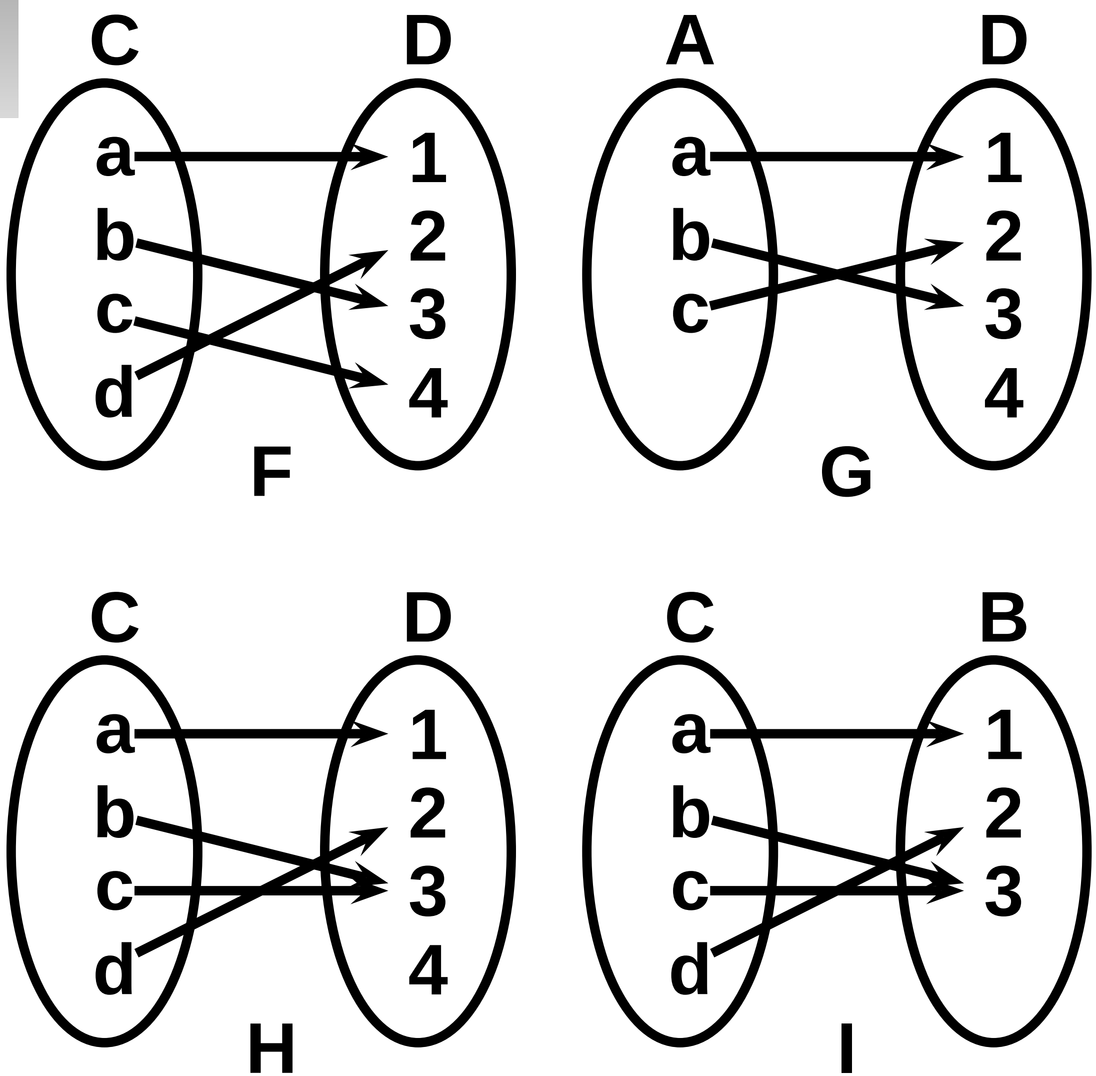
One-to-one and onto functions



F: 1-1, onto

G: 1-1

One-to-one and onto functions

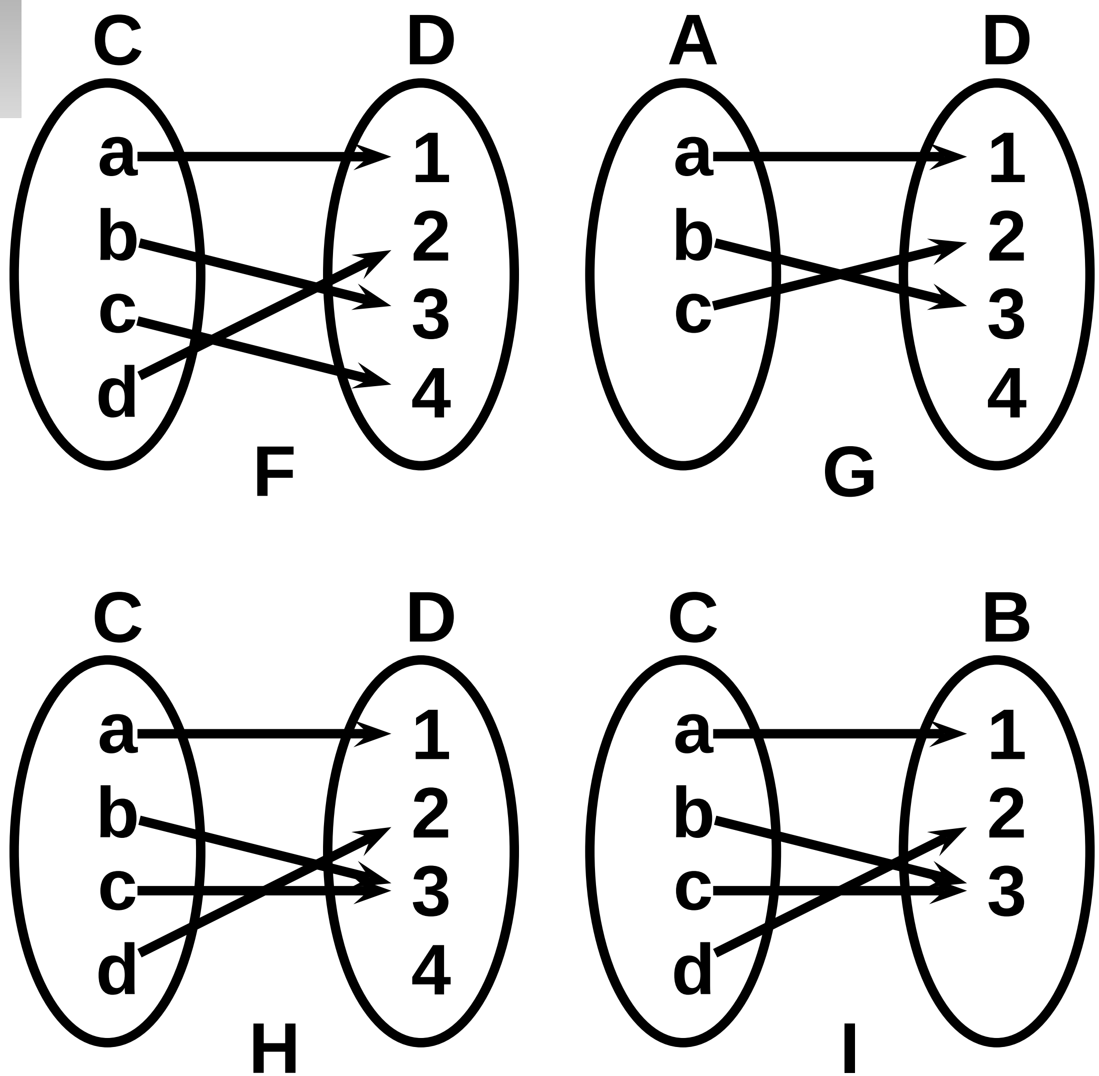


F: 1-1, onto

G: 1-1

H: neither

One-to-one and onto functions



F: 1-1, onto

G: 1-1

H: neither

I: onto

Some properties of Relations

Reflexive pairs

- We call a pair of the form

$$\langle x, x\rangle$$

  a **reflexive** pair.
- Note that some relations have reflexive pairs:

$$\langle t, t\rangle \in \text{same-place}$$

- Some do not.

$$\langle t, t\rangle \notin \text{SoundPlace}$$

Reflexivity

Definition 6. *Reflexivity of a relation*
A relation R is reflexive if and only if

$$\forall x \in Dom(R)\ \langle x,\, x\rangle \in R$$

This introduces the symbol $\forall$ (upside-down A). $\forall x$ should be read “for all x”.

for all x such that $x \in \mathrm{Dom}(\mathrm{R})\langle x,\, x\rangle \in \mathrm{R}$

$\forall x$ is always followed by some statement of a condition that all x’s satisfy. Usually we restrict $\forall x$ to particular set S of x’s: $\forall x \in$ S.

Examples of reflexivity

Example 10. *Reflexivity of* SamePlace
The SamePlace *relation is reflexive. Every sound is pronounced in the same place as itself.*

Example 11. *NonReflexivity of* SoundPlace
The SoundPlace *relation is not reflexive. No sound has itself as its place of articulation.*

Definition 7. *Irreflexivity of a relation*
A relation R is irreflexive if and only if

$$\forall x \in Dom(R)\ \langle x,\, x \rangle \notin R$$

Example of irreflexivity

Example 12. *Irreflexivity of* SoundPlace
The SoundPlace *relation is irreflexive. No sound has itself as its place of articulation. Observation: Any relation from*

sets A to set B is trivially irreflexive when A and B are disjoint.

Neither Reflexive nor Irreflexive

Example 13. *sum-of-factors (perfect numbers)*
Consider the function sum-factors.

$$\textit{sum-factors}(x) = y$$

if and only if the sum of all the prime numbers that evenly divide x, excluding x equals y.

$\textit{sum-factors}(2) = 1$ *1 is the only prime factor of 2*

$\textit{sum-factors}(4) = 3$ *1 and 2 are 4's prime factors and 1+2=3*

Sum-factors is not reflexive. Is it irreflexive? No, because the prime factors of 6 are 1, 2 and 3 and

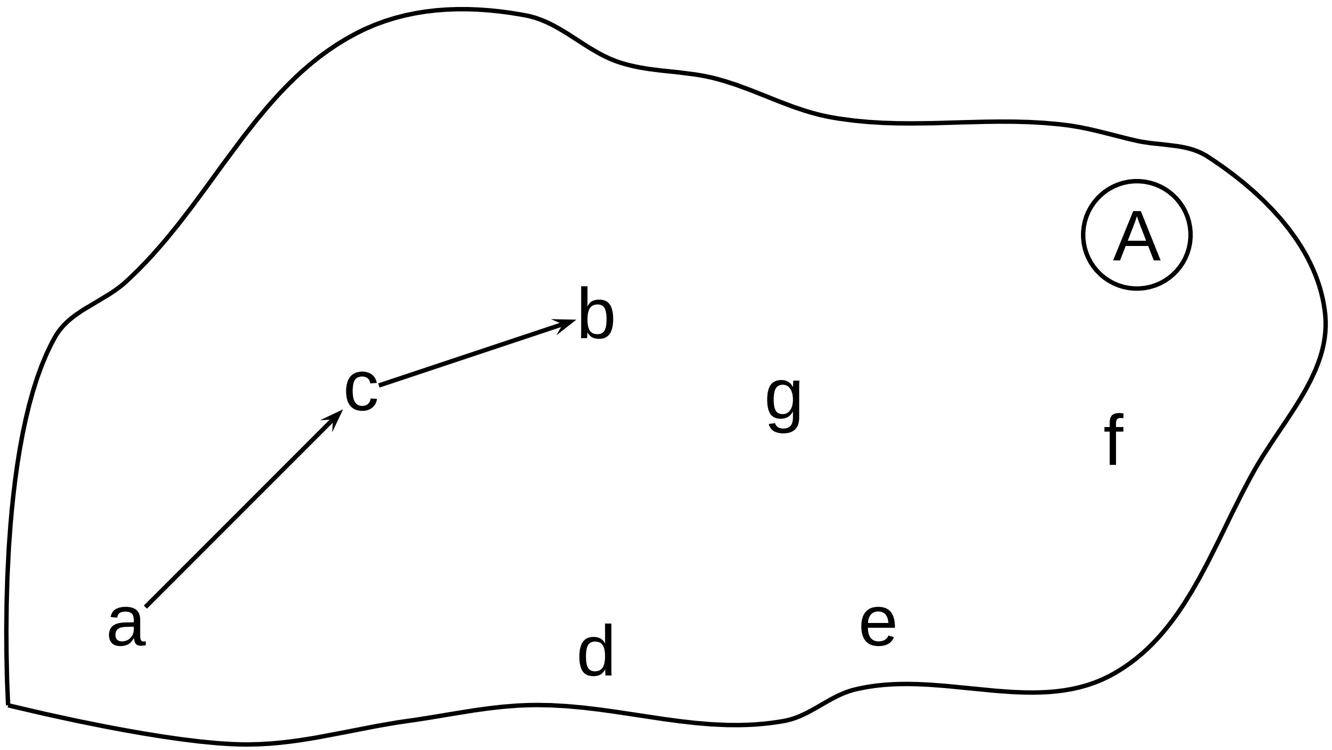
$$\textit{sum-factors}(6) = 6 \quad 6 = 1 + 2 + 3$$

Transitivity

Definition 8. *Transitivity of a relation*
A relation R is transitive if and only if

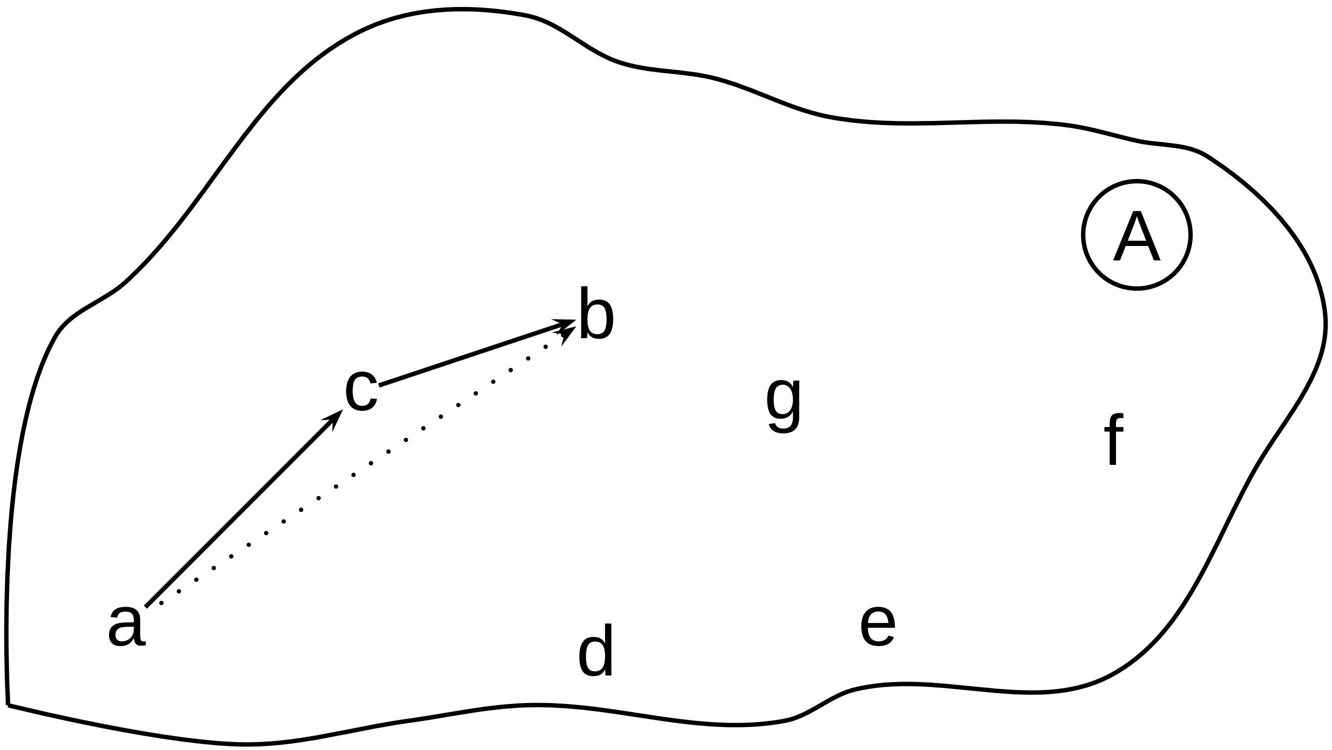
$$\forall \langle x,\, y \rangle \in R \textit{ If } \exists z \langle y,\, z \rangle \in R \textit{ then } \langle x,\, z \rangle \in R$$

Transitivity: The intuition



Solid links give a partial picture of a relation R from A to A. If R is transitive, then when the solid links exist, the dashed links must also exist.

Transitivity: The intuition



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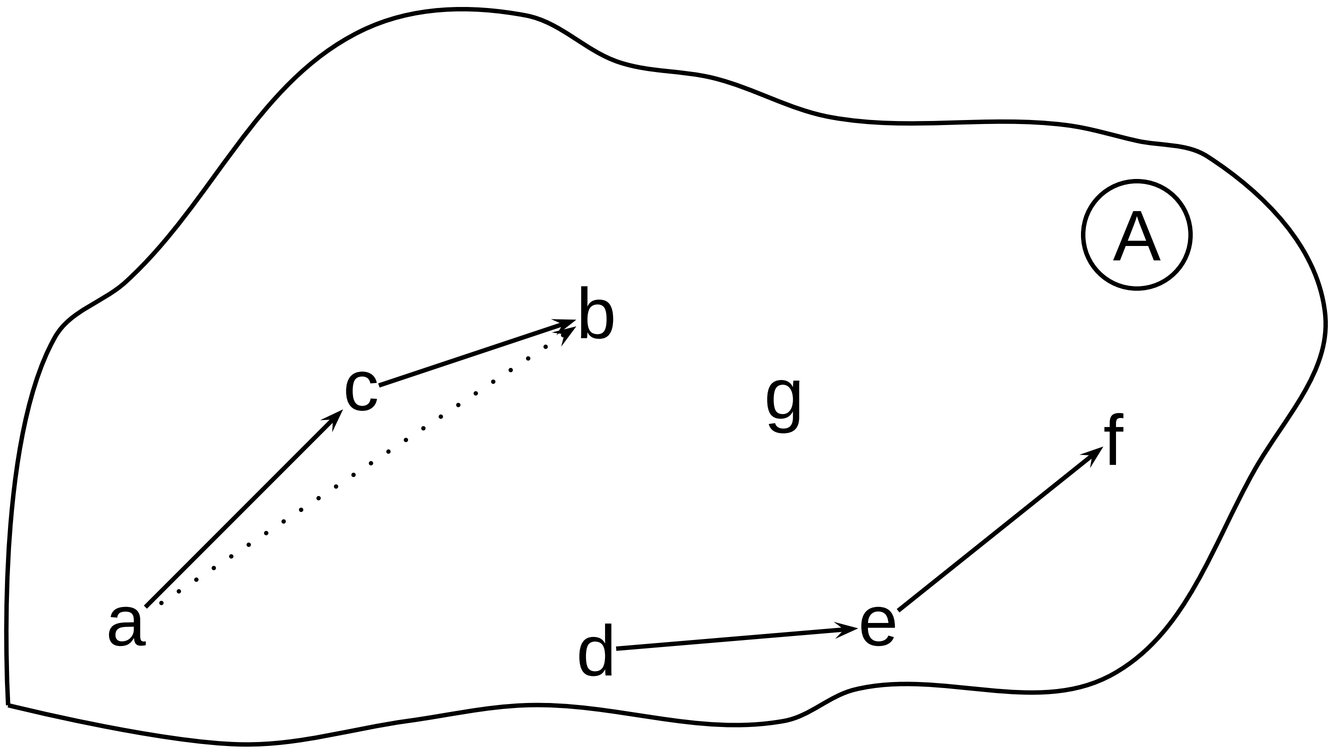

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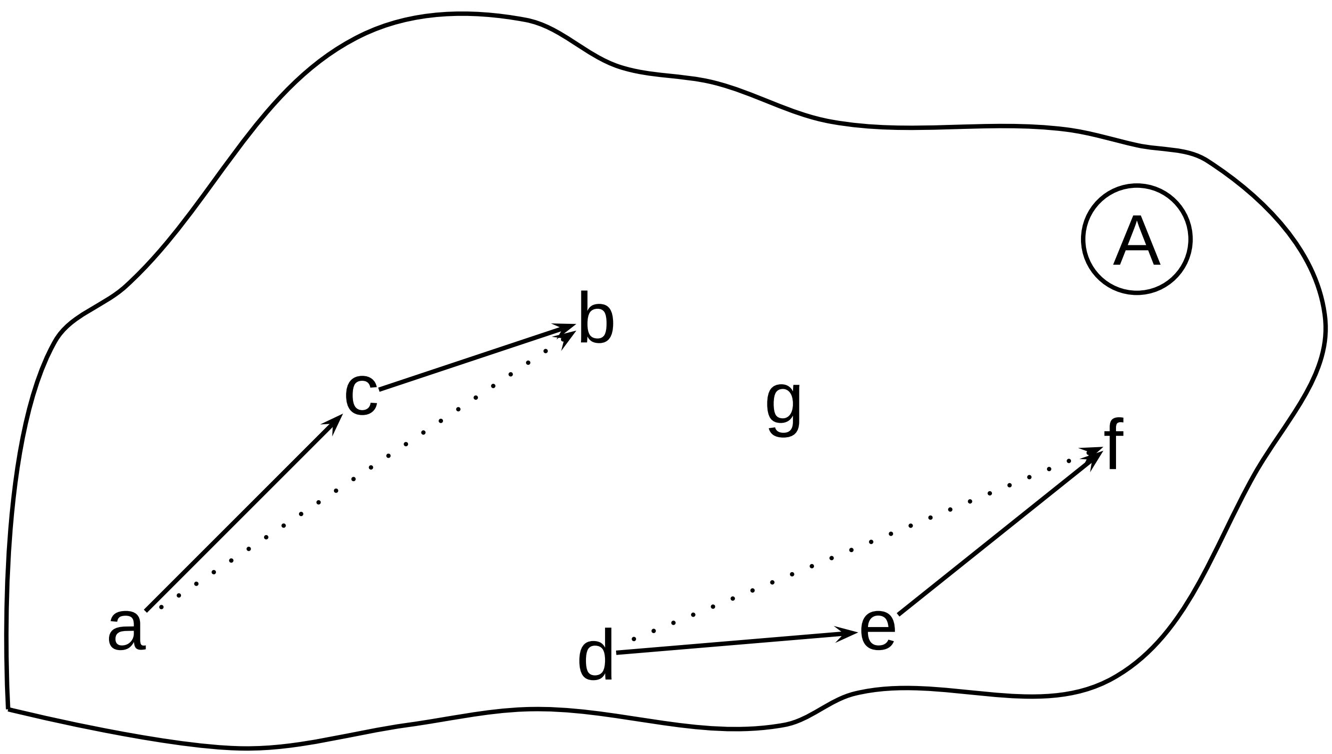

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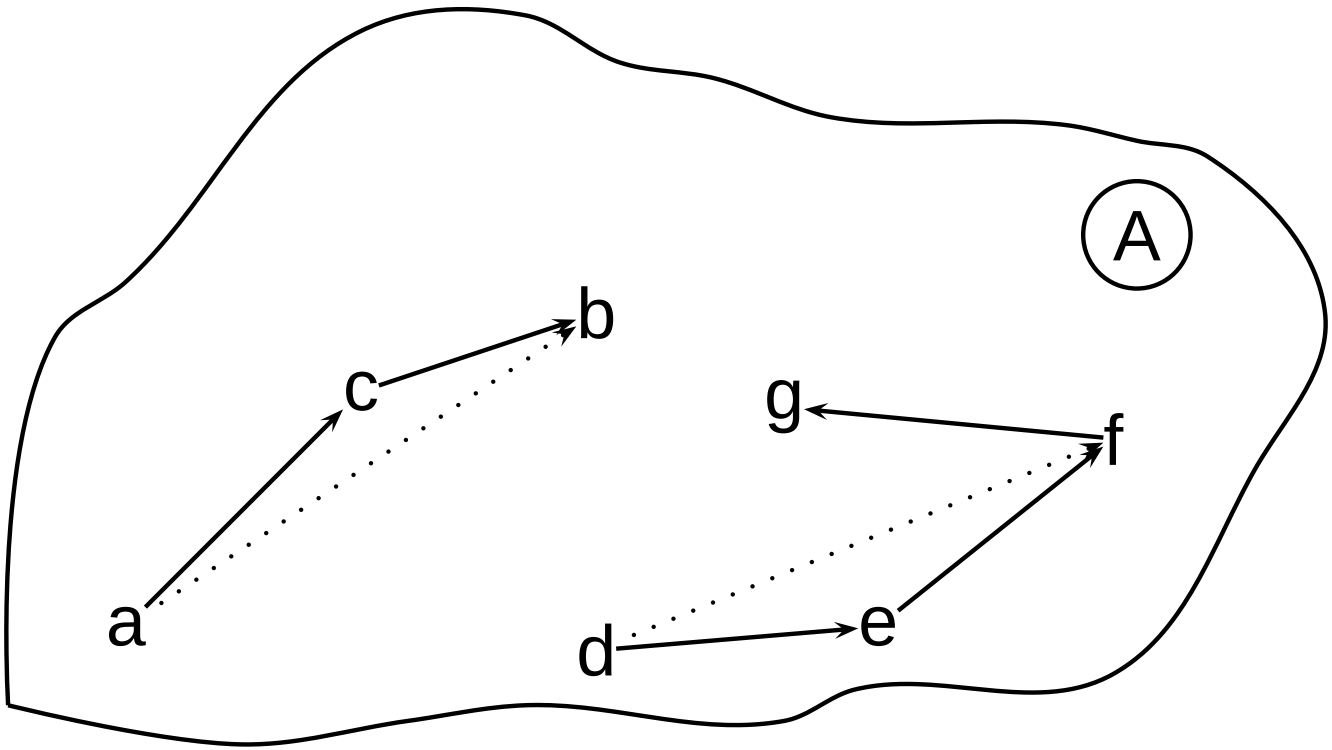

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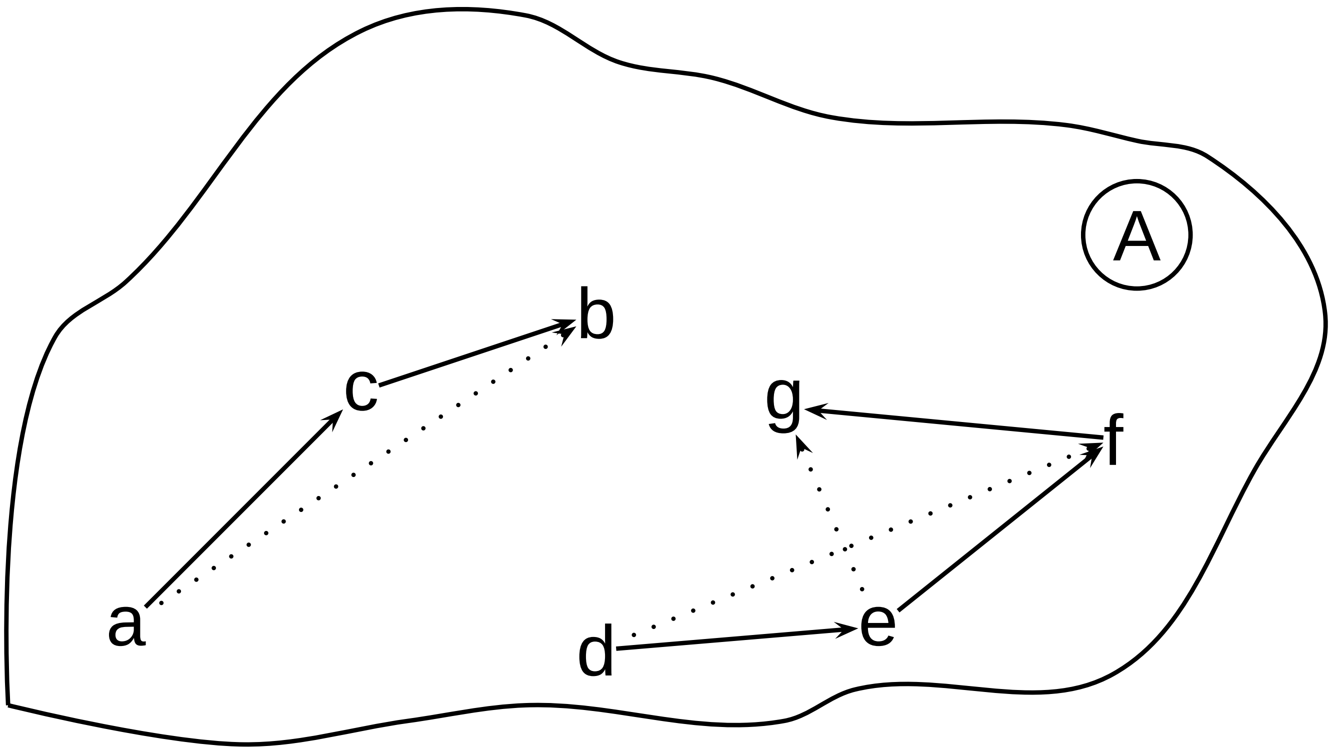

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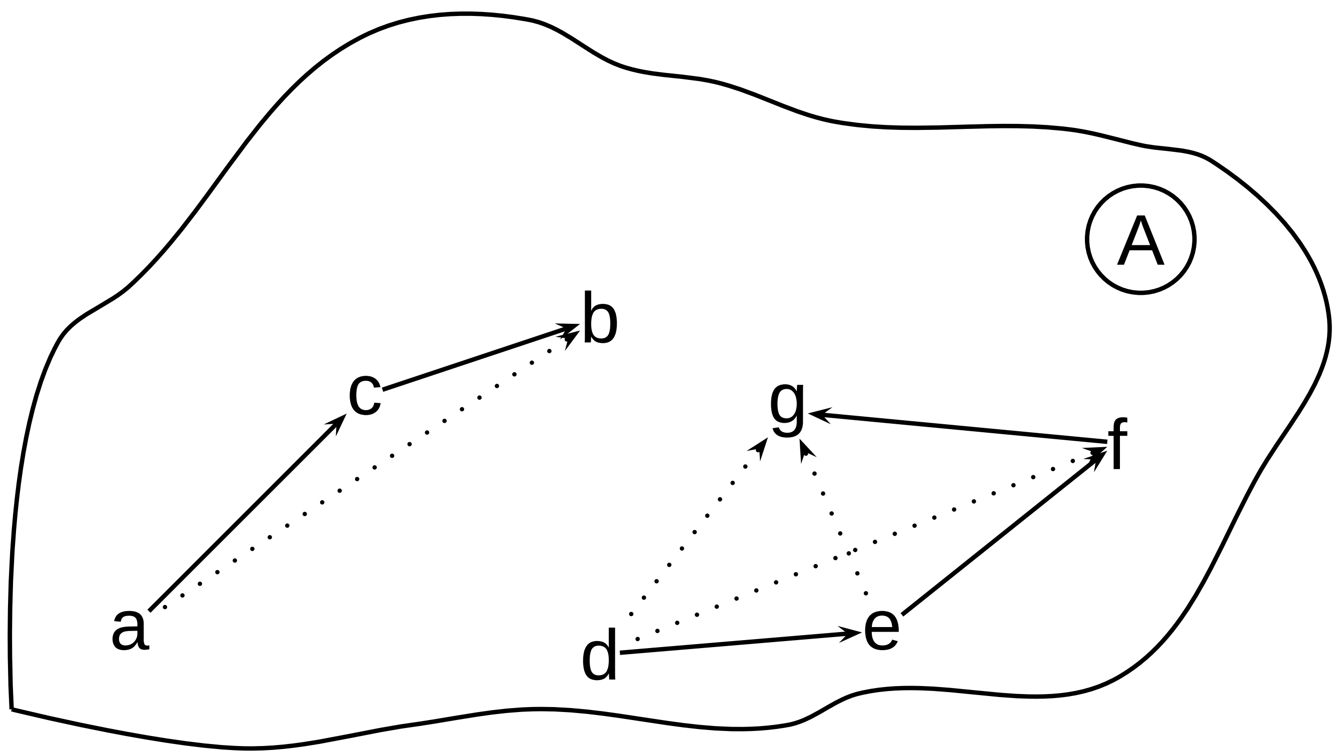

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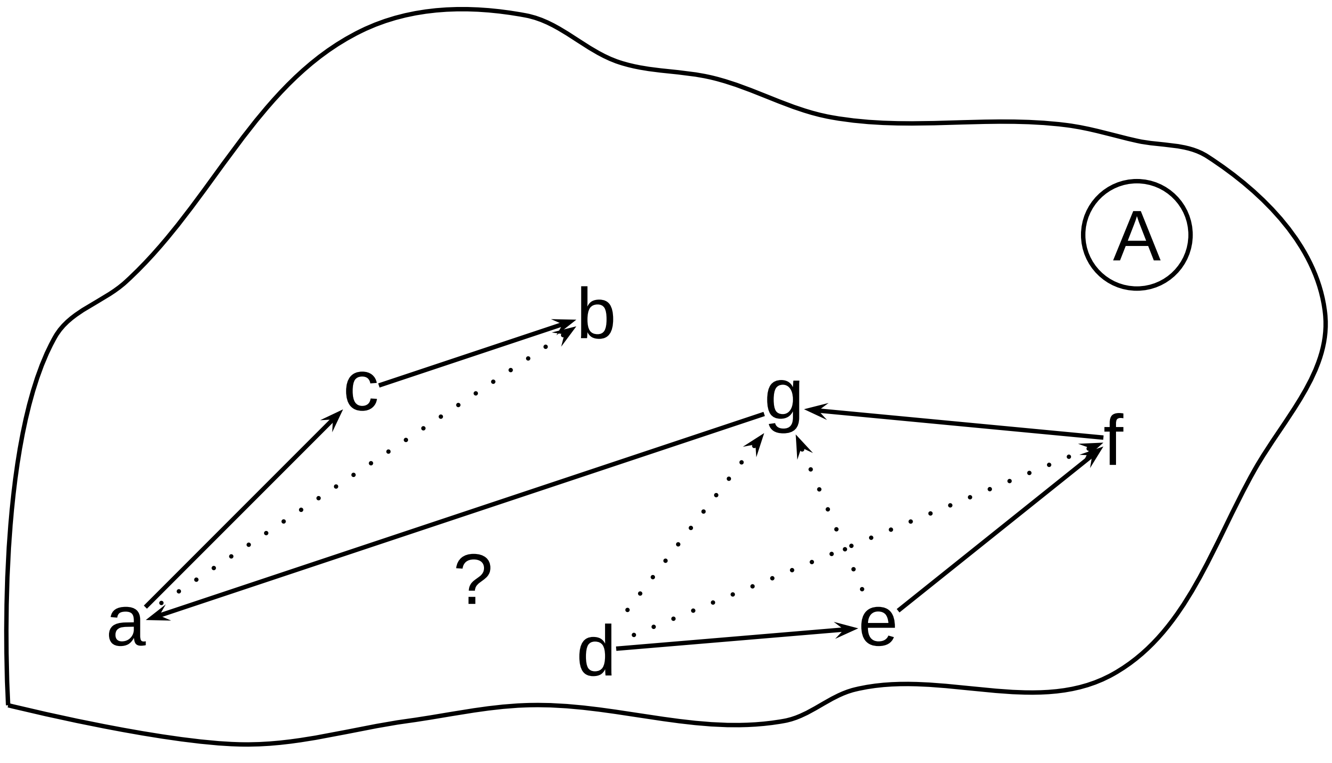

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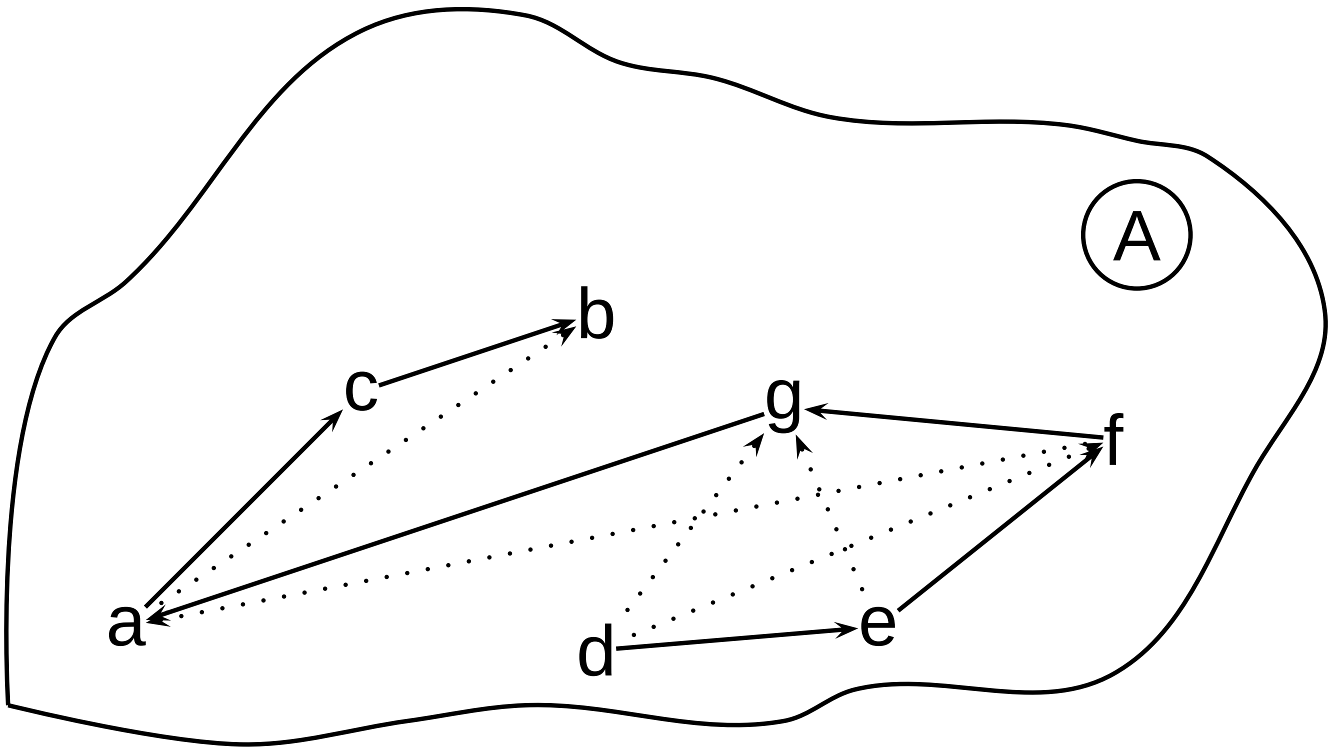

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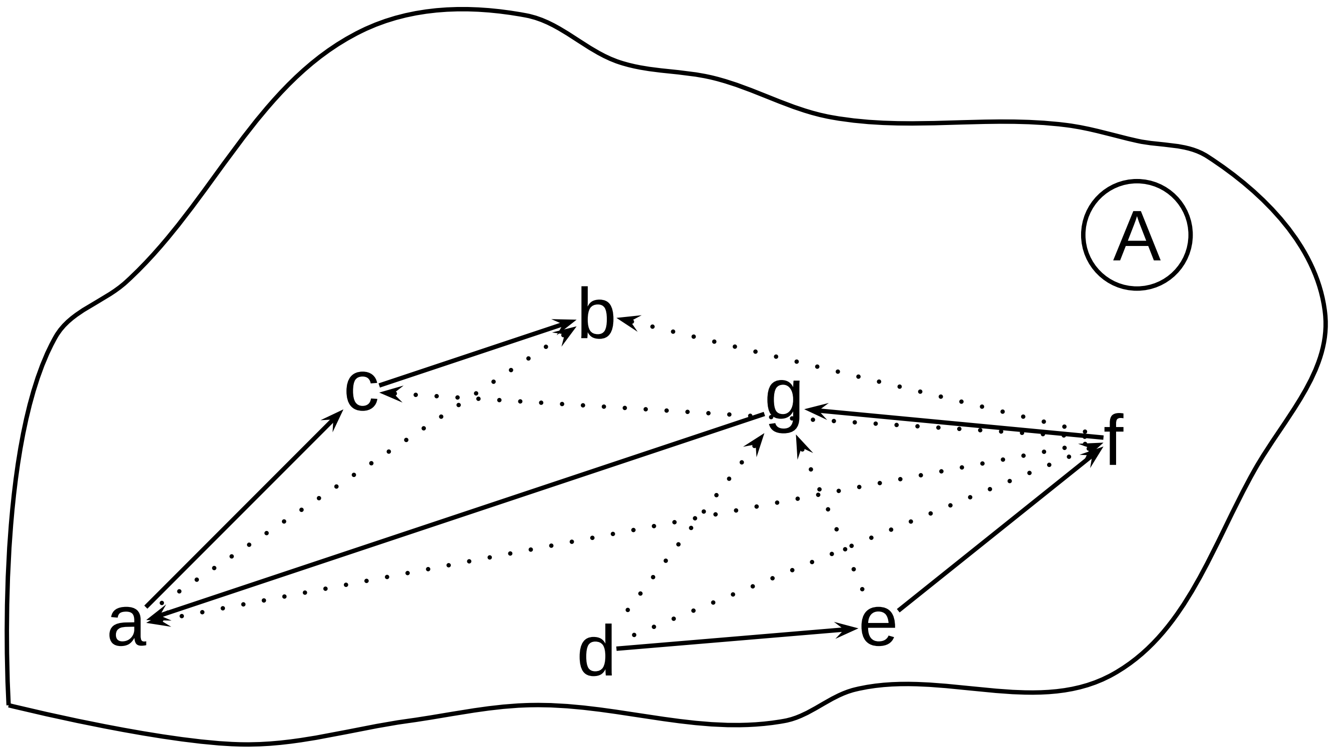

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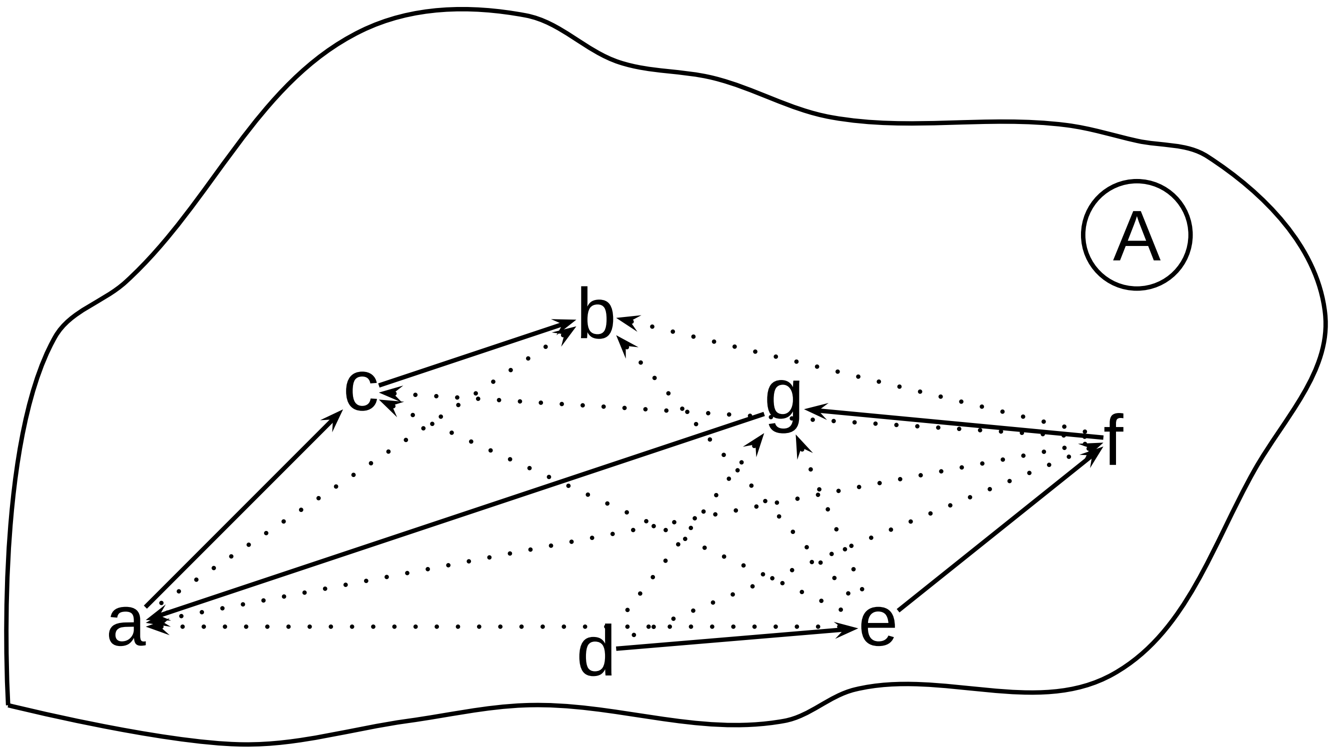

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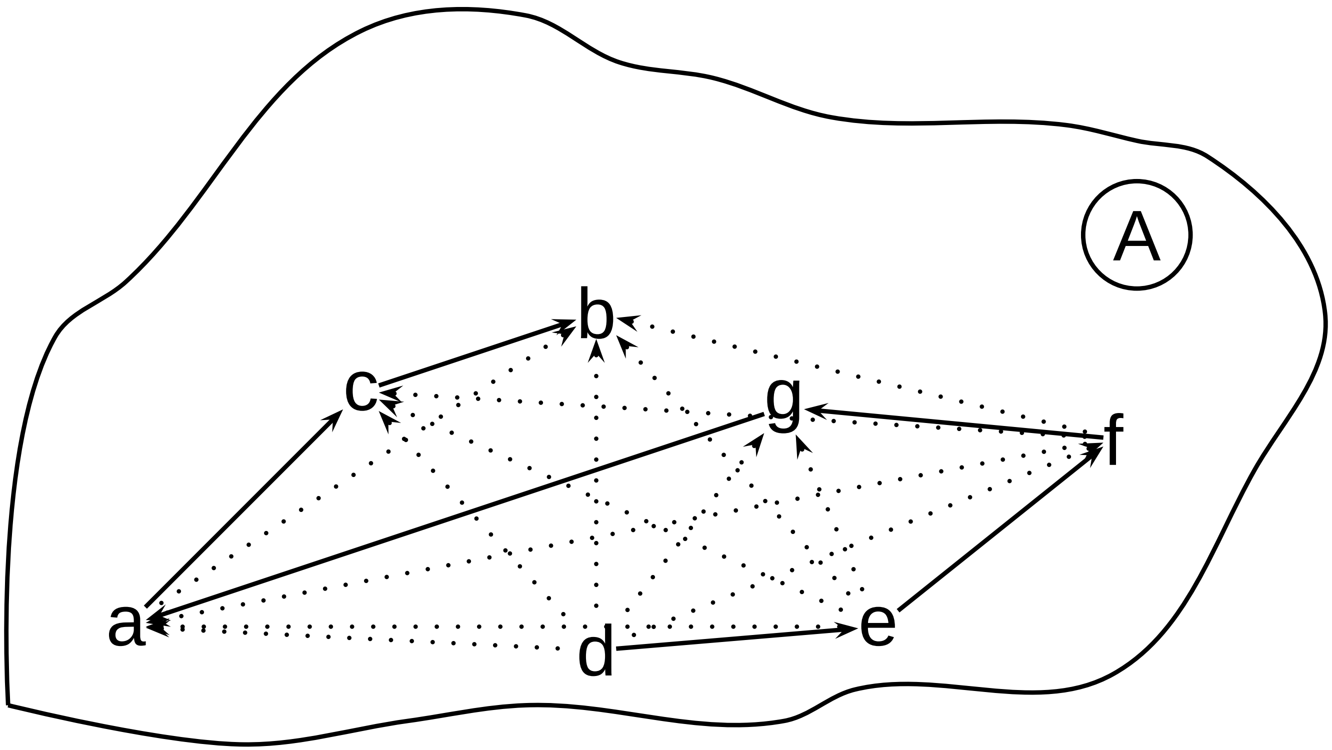

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Examples of transitivity

- The greater-then relation ($<$) on integers is transitive:

$$x < y$$
$$y < z$$
$$\Rightarrow$$
$$x < z$$

- The father-of relation is NOT transitive

  Fred is Tom’s father

  Tom is Hank’s father

  $\not\Rightarrow$

  Fred is Hank’s father

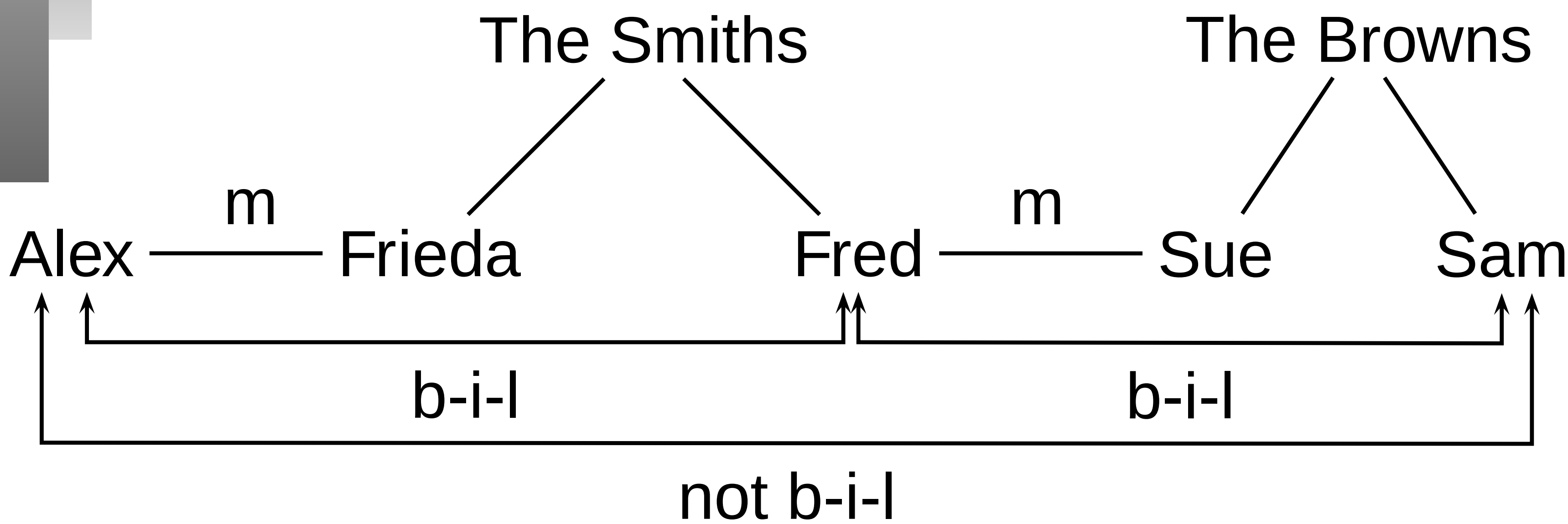
  In fact. Fred is Hank’s grandfather

More Transitivity examples

Example 14. *Transitivity of brother-of relation.*
If Fred is Sam's brother and Alex is is Fred's brother, then Alex is Sam's brother.

Example 15. *Non-Transitivity of brother-in-law relation.*
If Fred is Sam's brother-in-law (because Fred is married to Sam's sister Sue) and Alex is Fred's brother-in-law (say, because Alex is married to Fred's sister Frieda), then Alex is not *Sam's brother-in-law.*

Non-transitivity of brother-in-law relation



Non-transitivity of the *brother-in-law* relation. Frieda and Fred are syblings, as are Sue and Sam. Marriage links are labeled *m*; brother-in-law links are labeled *b-i-l*.

Definition 9. *Symmetry of a relation*
A relation R is symmetric if and only if

$$\forall \langle x, y \rangle \in R \, [\langle y, x \rangle \in R]$$

Symmetry examples

Example 16. *Symmetry of sybling relation*
If Fred is Sue's sybling then Sue is Fred's sybling.

Example 17. *Symmetry of cousin relation*
If Fred is Sue's cousin then Sue is Fred's cousin.

Example 18. *Non-symmetry of brother relation*
If Fred is Sue's brother and Sue is female it is not the case that Sue is Fred's brother.

Asymmetry

Definition 10. *Asymmetry of a relation*
A relation R is asymmetric if and only if

$$\forall \langle x, y \rangle \in R \, [\langle y, x \rangle \notin R]$$

Example 19. *Asymmetry of father relation.*
If Fred is Bob's father it is definitely not the case that Bob is Fred's father.

Example 20. *Symmetry and the brother relation.*
If Fred is someone's brother they may or may not be his brother, depending on their sex.