Equivalence and Classification

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Equivalence Relations

R: an Equivalence Relation

Definition 1. *A relation R in the set A is an equivalence relation in A if and only if*

1. *R is reflexive on A; and*
2. *R is symmetric; and*
3. *R is transitive.*

Example I: identity

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- Symmetry: Equality is symmetric. If $x = y$, then $y = x$.
- Transitivity: If $x = y$ and $y = z$, then $x = z$.

Example II: A $\times$ A

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- Transitivity: If $\langle x, y\rangle \in$ A $\times$ A, and $\langle y, z\rangle \in$ A $\times$ A, this shows $x, z \in$ A. So $\langle x, z\rangle \in$ A $\times$ A.

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- Transitivity: Suppose x has the same voicing as y and y has the same voicing as z, then x has the same voicing as z.
  Proof: Suppose x is voiced. Then if x stands in the same voicing relation to y, then y is voiced. And if y stands in the same voicing relation to z, then z is voiced. So x and z have the same voicing. The case of x being voiceless is completely parallel.

Difference is divisible by 3

Consider this R

$$\mathsf{R} = \{\langle a,\, b\rangle \mid a - b \text{ is evenly divisible by 3}\,\}$$

Consider the case where $b = 7$.

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Consider the case where $b = 7$. A list of numbers a that stand in the relation R when $b = 7$:

a	1	4	7	10	13	16	$\ldots$
$a - 7$	-6	-3	0	3	6	$\ldots$	

Notice all the numbers in the top row stand in the relation R to each other as well as to 7, because they are all separated by multiples of 3. Is R an equivalence relation?

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- Reflexivity: $x - x = 0$ and 0 is divisible by 3.
- Symmetry: If $x - y$ is divisible by 3, then $y - x$ is divisible by 3.
- Transitivity: Proof left to you.

Parallel lines

Let L be the set of lines in some plane P. Let $\|$ be the relation that holds between two lines if and only if they are parallel or equal.

Is $\|$ an equivalence relation in L?

1. Reflexive: .
2. Symmetric:
3. Transitive:

What if we remove the restriction that the lines are coplanar?

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Is $\|$ an equivalence relation in L?

1. Reflexive: Yes, by definition.
2. Symmetric: Yes, if $x \parallel y$, then $y \parallel x$.
3. Transitive: Yes.

What if we remove the restriction that the lines are coplanar?

Same first ordinate

Let $\mathrm{S} \times \mathrm{T}$ be the cross product of two sets S and T.

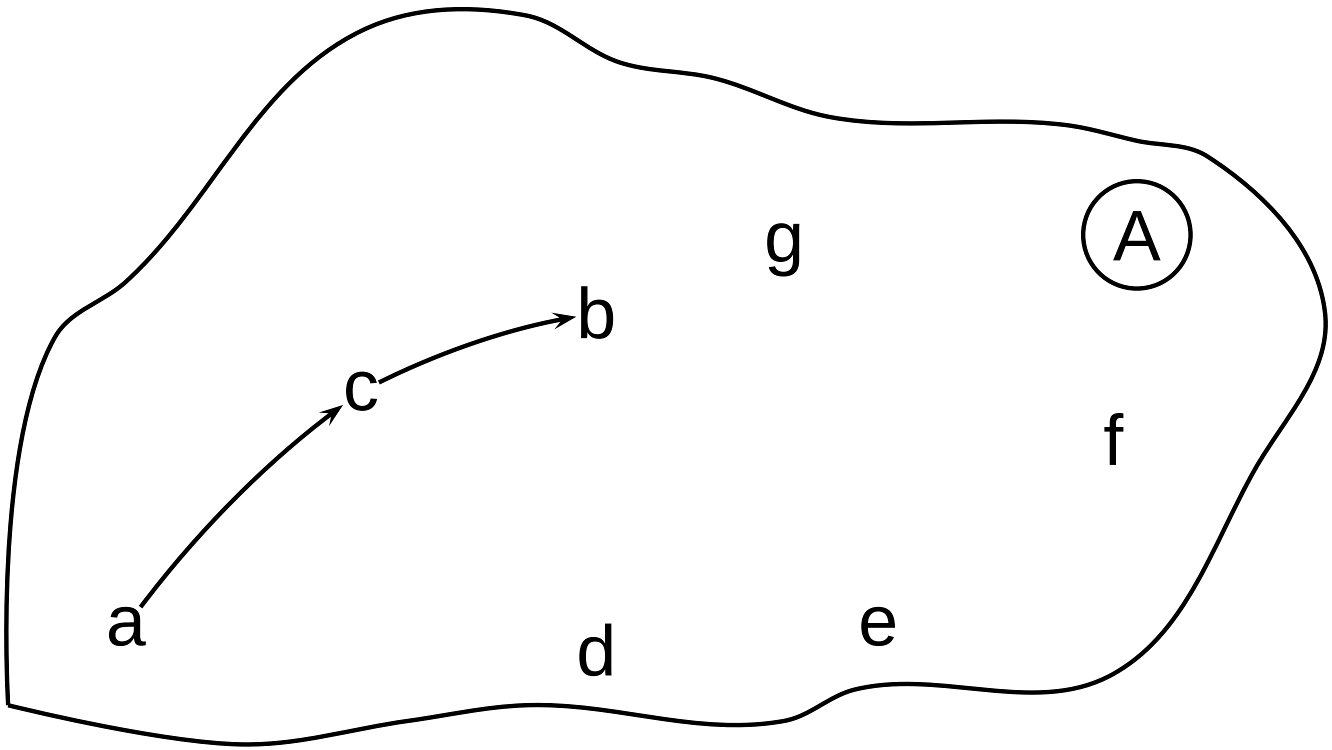
$$\langle x,\, y\rangle \mathsf{E} \langle x',\, y'\rangle \text{ iff } \langle x,\, y\rangle \in \mathrm{S} \times \mathrm{T} \text{ and } \langle x',\, y'\rangle \in \mathrm{S} \times \mathrm{T} \text{ and } x = x'.$$

Is E an equivalence relation?

Non-examples

- Sybling relation
- $\{\langle x, y \rangle \mid$
  height difference of x and y is less than 5 inches$\}$
- $\{\langle x, y \rangle \mid x \leq y\}$
- $\{\langle x, y \rangle \mid$
  x and y are people who eat at the same restaurant$\}$

Equivalence relations: The intuition



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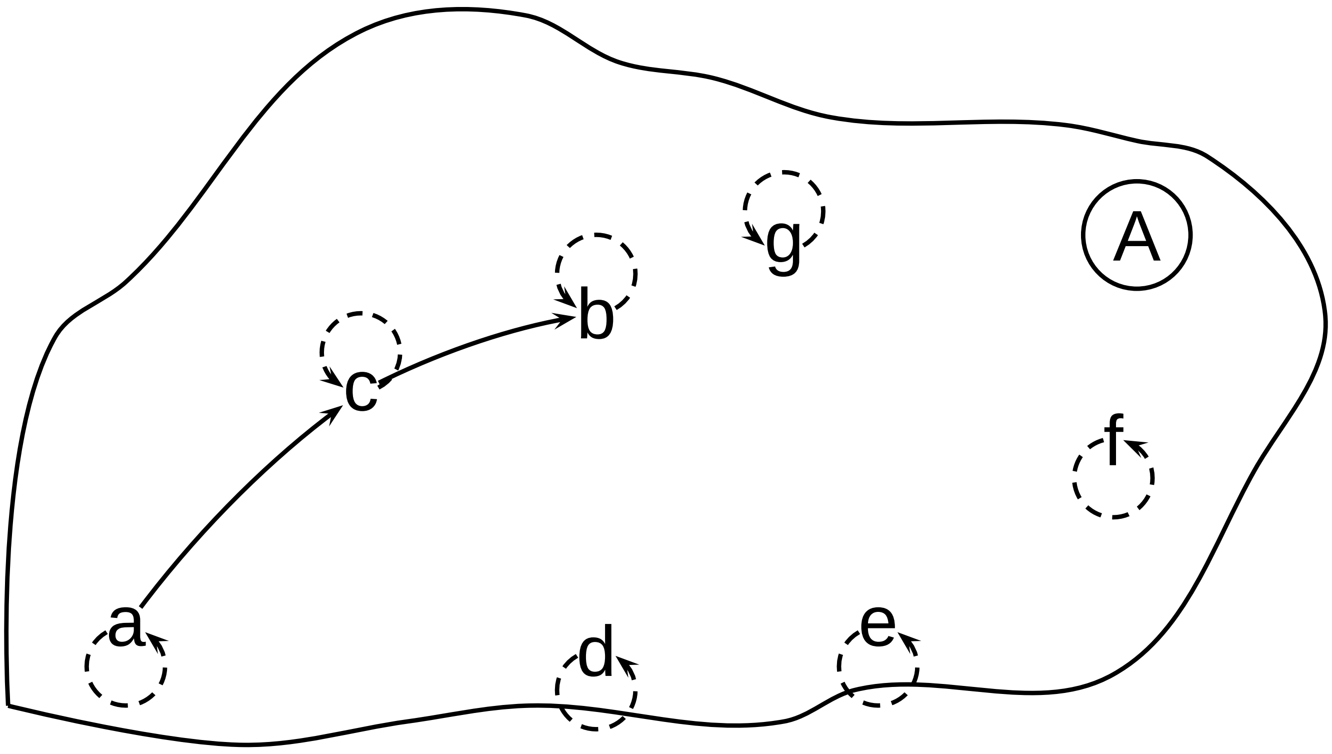

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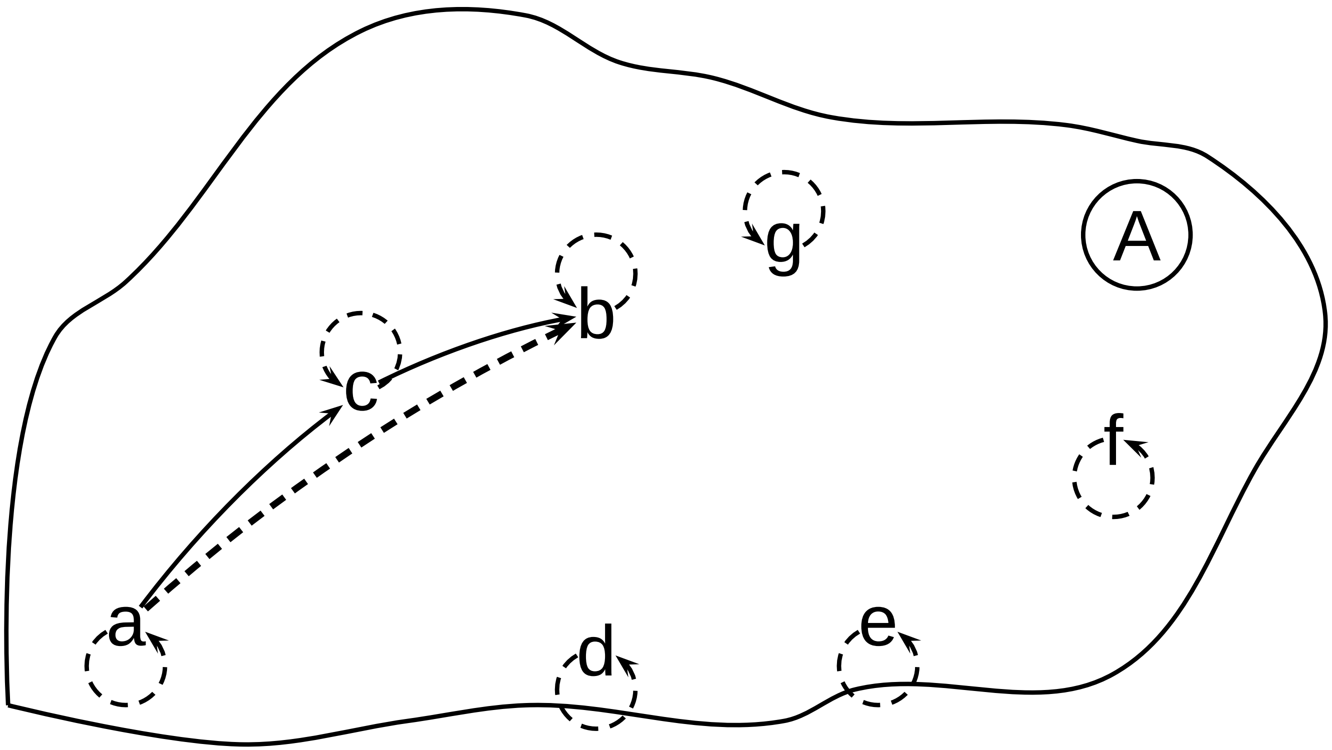

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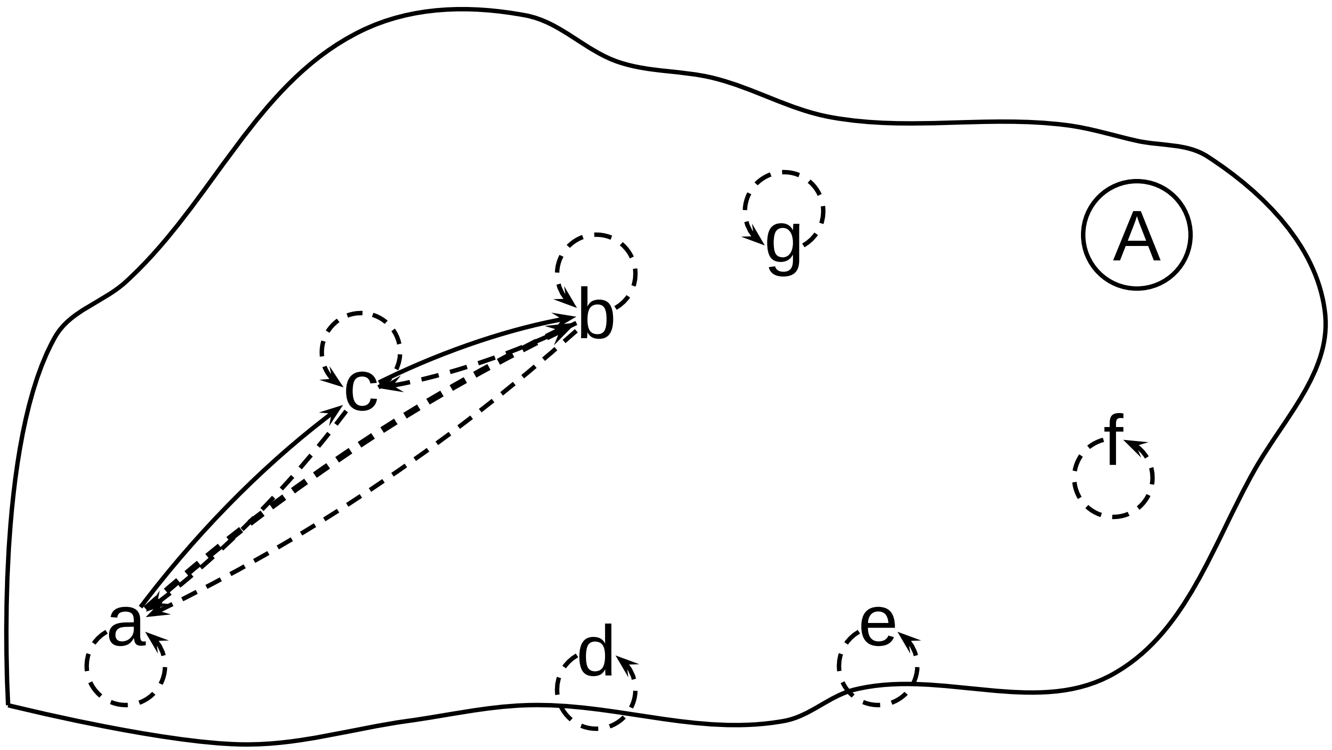

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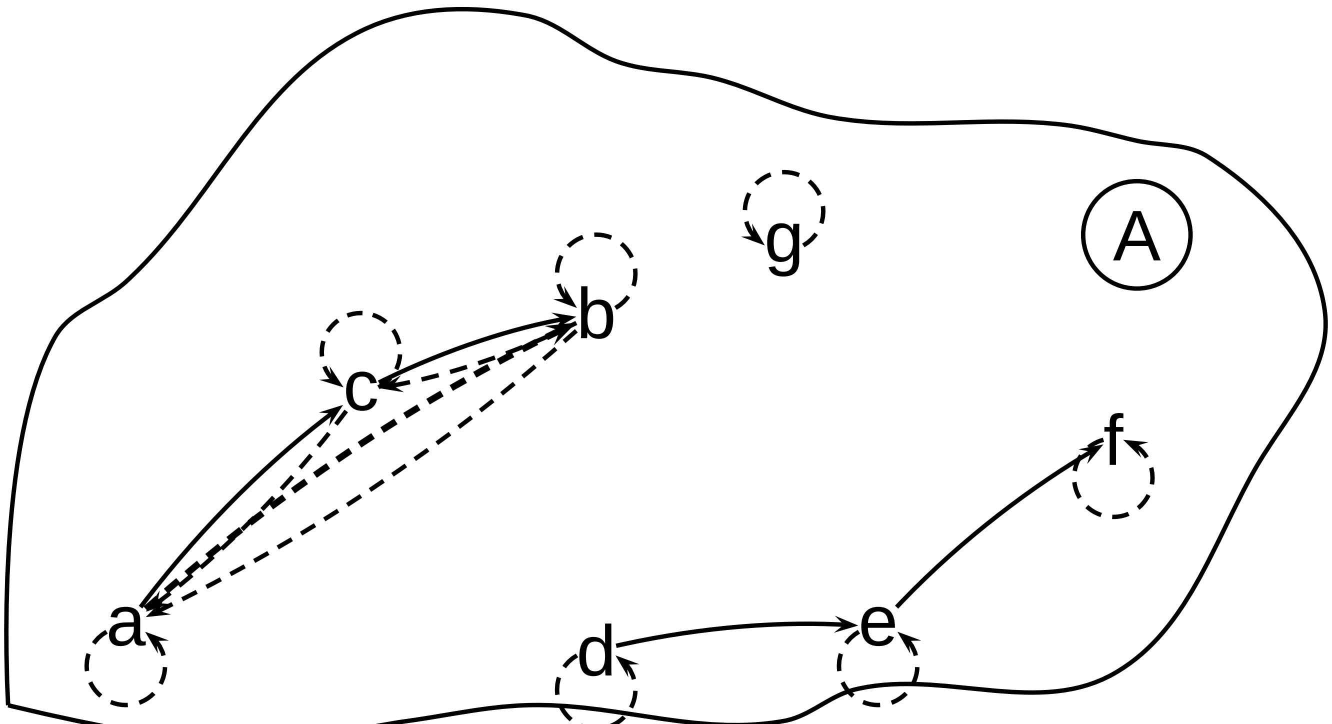

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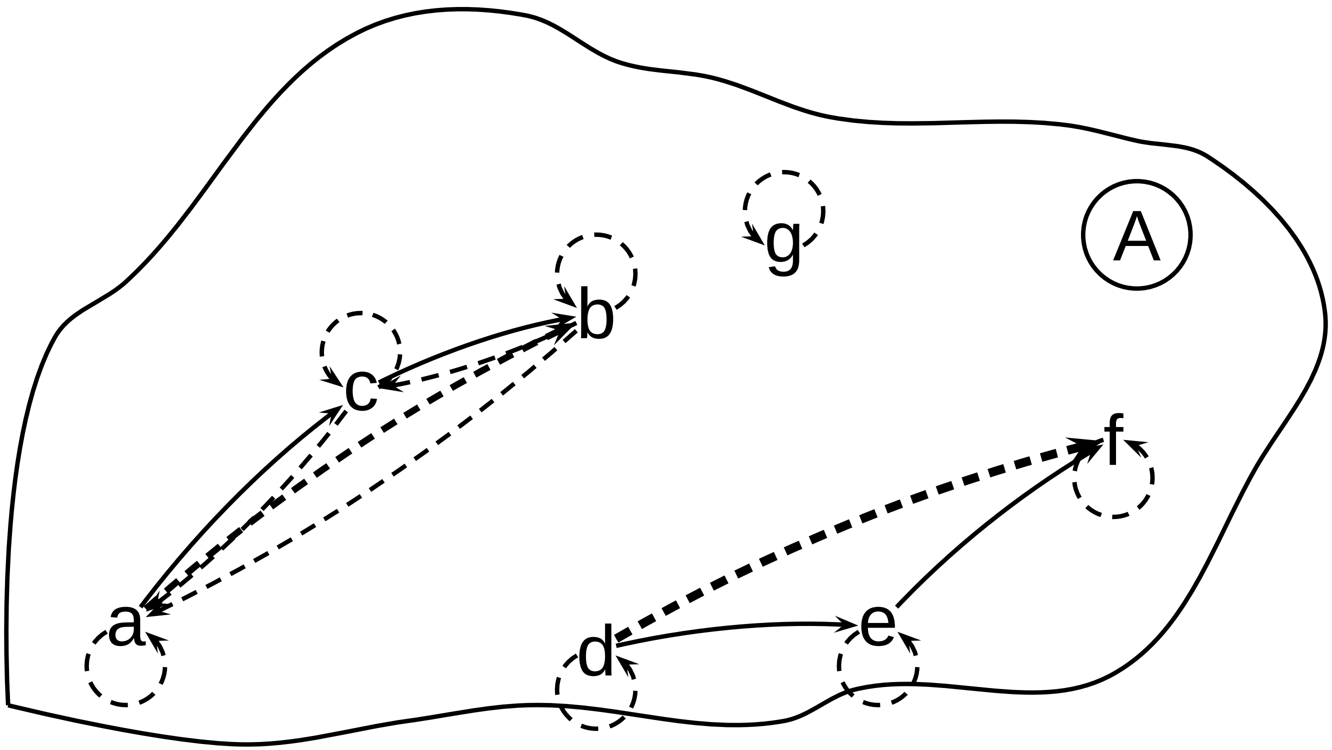

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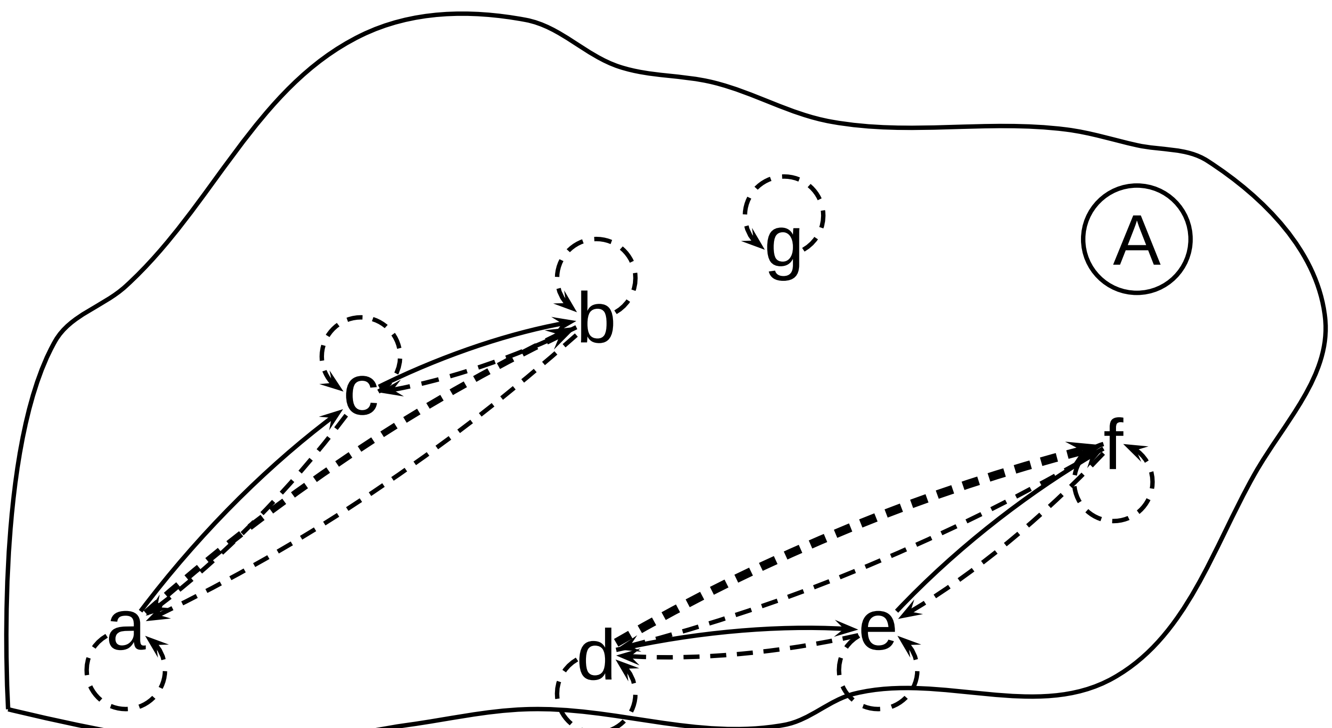

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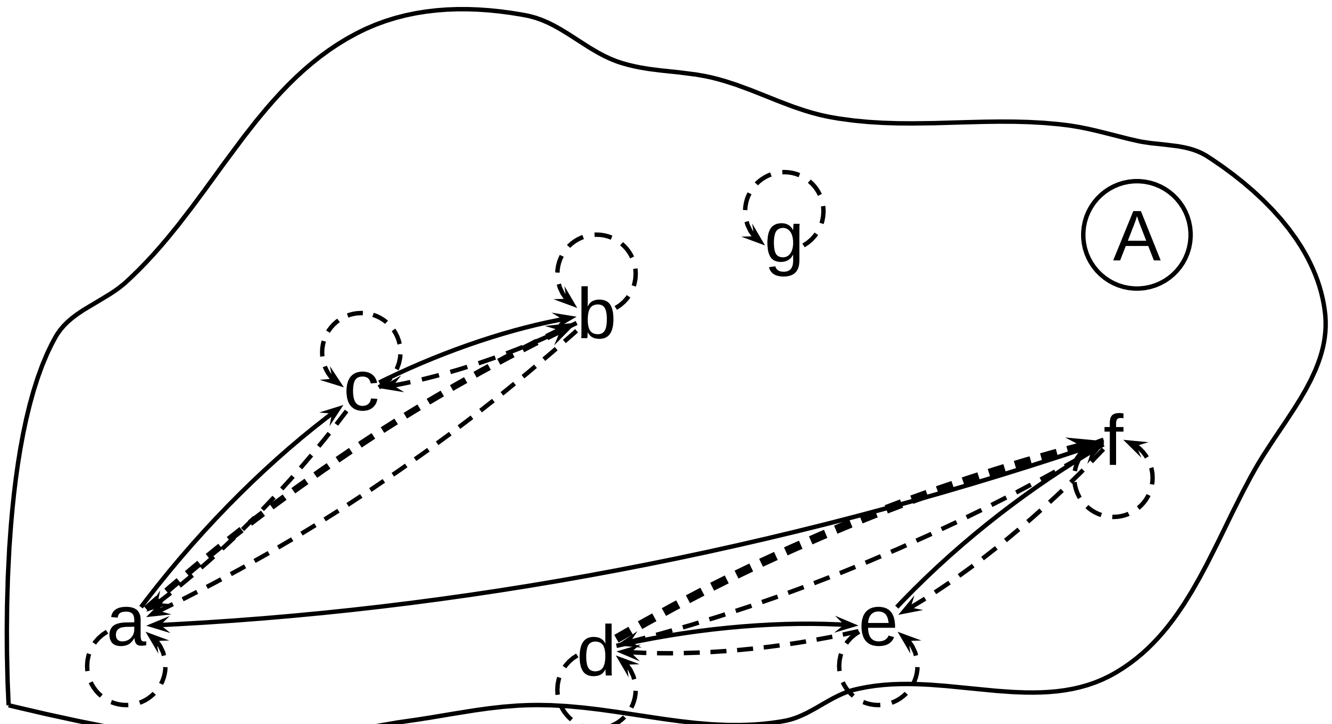

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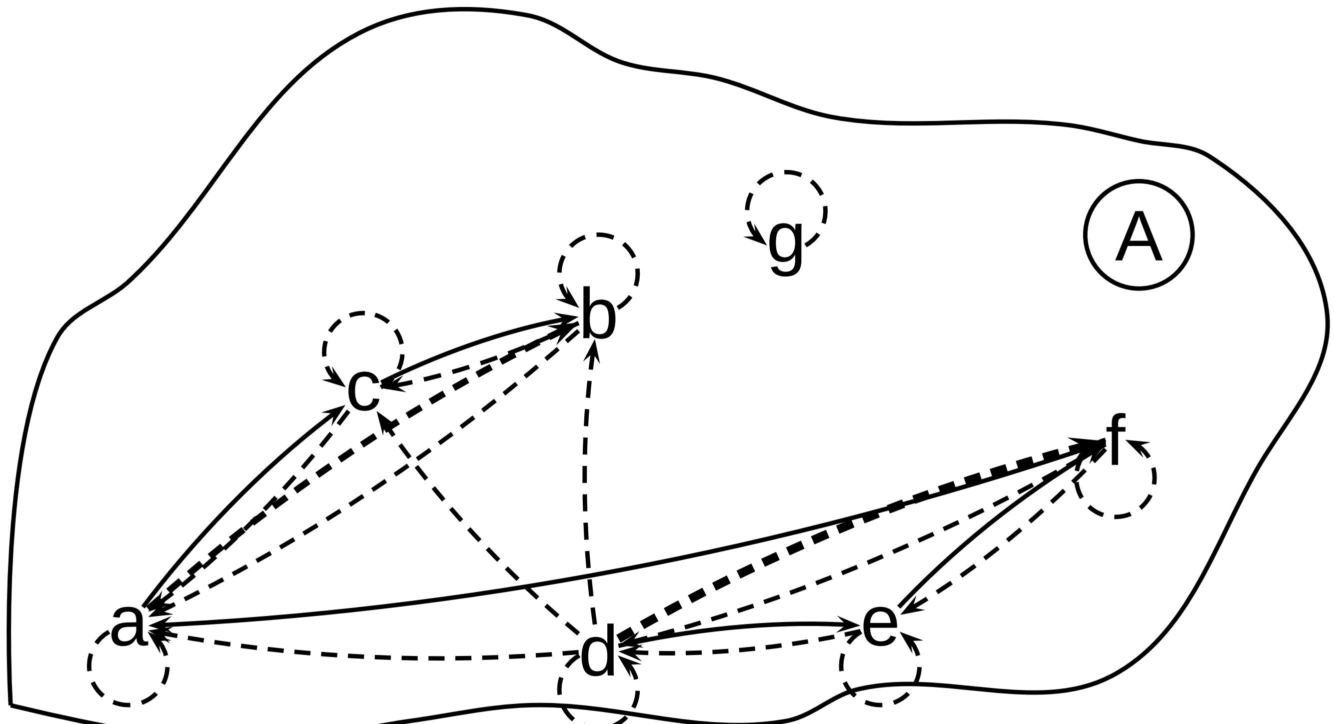

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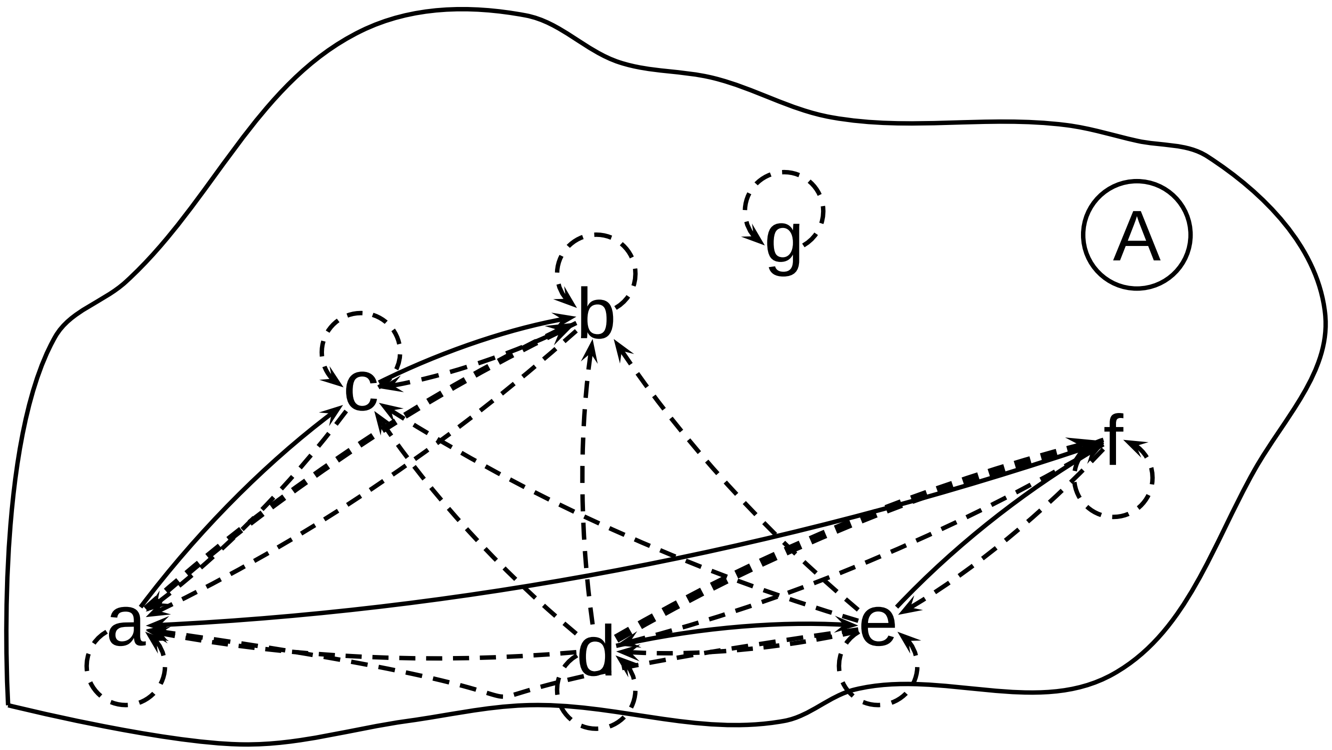

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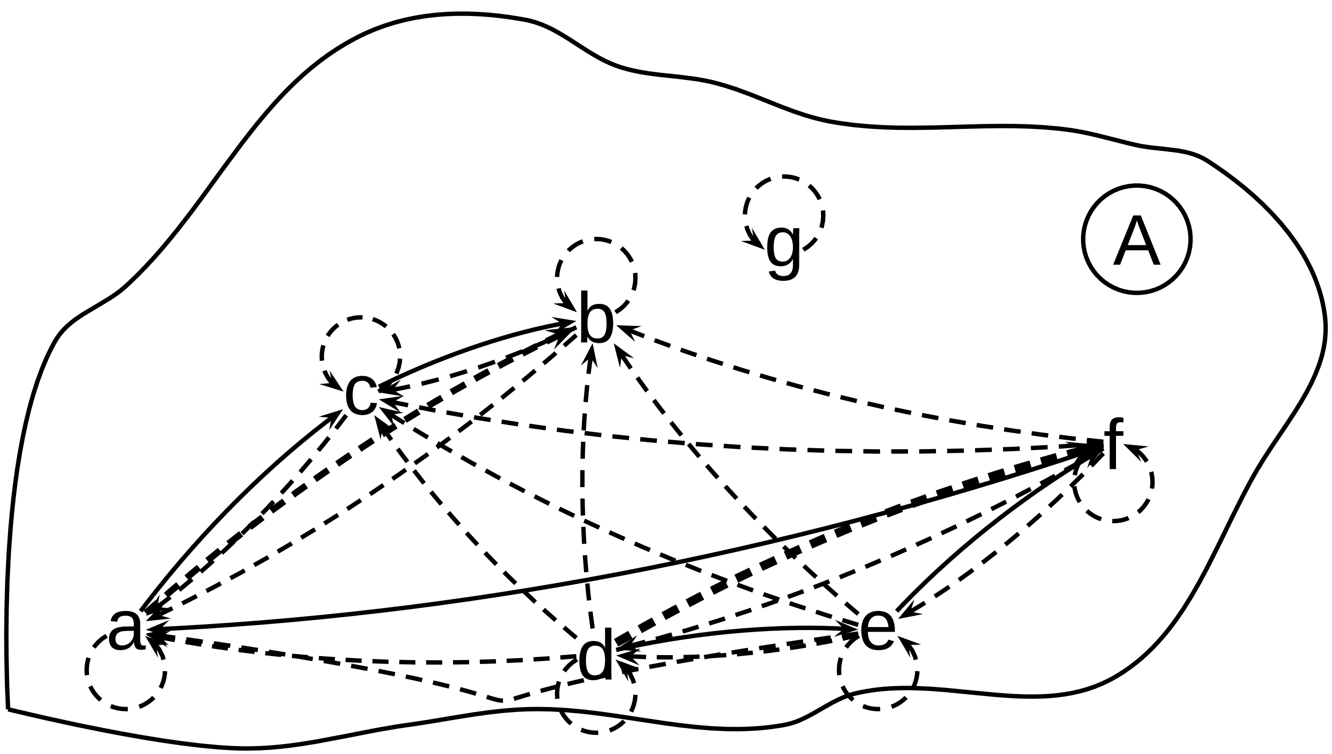

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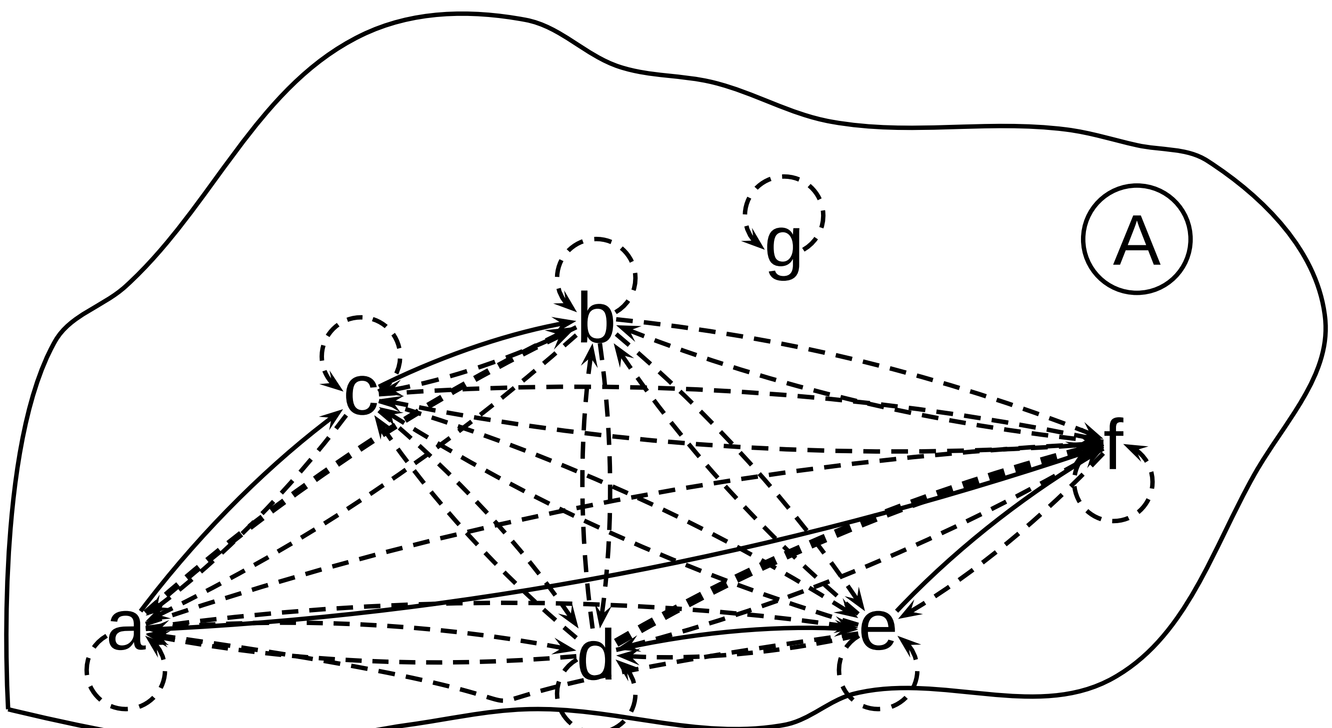

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Equivalence relations: Partitions

Definition 2. *A partitioning of a set A is a set of disjoint sets* $p_1,\ p_2,\ \ldots p_n$ *such that*

$$A = p_1 \cup p_2 \cup p_3 \cup \ldots p_n$$

Definition 3. *An equivalence relation R on A defines a partitioning of R. Each of the sets in the partition is called an* equivalence class.

We define π_{R}, *the partition induced by equivalence relation R, as follows.*

$$\pi_{\mathrm{R}} = \{ S \mid \exists x \in A[\mathrm{S} = \{ y \mid \langle x,\ y \rangle \in R \}] \}$$

Partitions: The two extreme cases

- $|\pi| = |A|$. The identity relation on a set $A = \{a, b, c\}$ induces the partition:

$$\pi_{\mathrm{I}} = \{\{a\}, \{b\}, \{c\}\}$$

- $|\pi| = 1$. If $A = \{a, b, c\}$, then $A \times A$ induces the partition:

$$\pi_{A \times A} = \{\{a, b, c\}\}$$

- Exercise: List the other possible partitions of A. Give the relations that induce them.

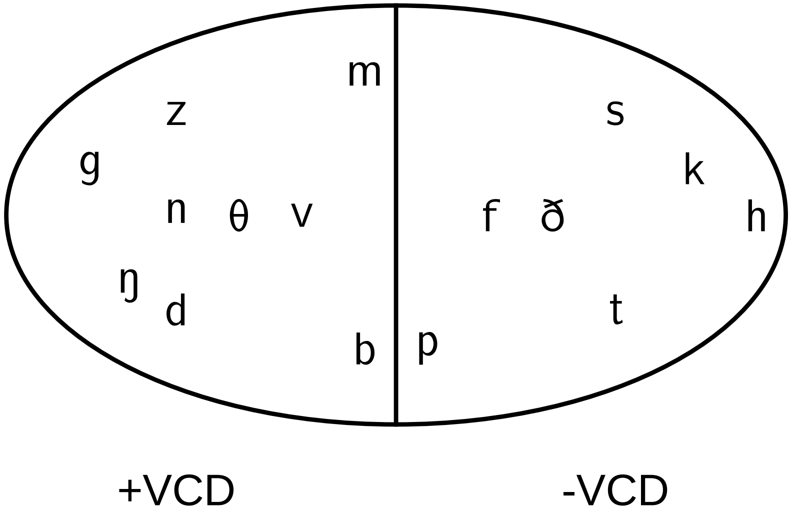
The **difference is divisible by 3** relation partitions the set of integers into three sets we'll call S_0, S_1, and S_2:

$$S_0 = \{0, 3, 6, 9, \ldots\}$$
$$S_1 = \{1, 4, 7, 10, \ldots\}$$
$$S_2 = \{2, 5, 8, 11, \ldots\}$$

Note we have 3 disjoint **infinite** sets which unioned together give us the entire set of integers.

SameVoicing

We call the partition induced by the SameVoicing relation **Voice**



SamePlace

We will call the partition induced by the SamePlace relation **Place**:

p			t		k	
b			d		g	
m			n		ŋ	
	f					
	v					
		θ	s	ʃ		h
		ð	z	ʒ		

Definition 4. *A* **feature space** *is a pair of a set A together with a set* Π *of partitions of A. A is called the domain of the feature space. Each member of* Π *is called a* **feature**.

Example 1. *Let us choose the set of obstruents as our domain, and the partitions induced by the SameVoice relation and the SamePlace relation as our features. Then*

$$\mathit{PhonFeatures} = \langle \mathit{Obstruents}, \{\, \text{Place}, \text{SameVoice} \,\} \rangle$$

is a feature space with Obstruents as its domain and

$$\{\, \text{Place}, \text{Voice} \,\}$$

as its features.

Feature Specification

Definition 5. *A* **feature specification** ξ *chosen from a feature space F is a set S such that each member of* ξ *is a* **member** *of one of the features of F.*
If

$$\xi = \{ \; Val1, Val2 \} \quad \textit{where} \quad Val1 \in Feat1, Val2 \in Feat2$$

in feature space

$$F = \langle A, \{Feat1, \; Feat2\} \rangle,$$

then feature specification ξ *can be written*

$$\xi = \begin{bmatrix} Feat1 & Val1 \\ Feat2 & Val2 \end{bmatrix}$$

We call $[\![\xi]\!]$ **the denotation** *of a feature specification, defined as the intersection of its partitions:*

$$[\![\xi]\!] = Val1 \cap Val2$$

Feature Specs in PhonFeatures

Example 2. *Let:*

$$\begin{aligned}
\textit{PhonFeatures} &= \langle \textit{Obstruents}, \{\text{Place}, \text{Voice}\}\rangle \\
\textit{Voice} &= \{ \textit{Plus}, \textit{Minus} \} \\
\textit{Place} &= \{ \textit{Labial}, \textit{Labiovelar}, \textit{Interdental}, \textit{Alveolar} \\
&\quad \textit{Alveopalatal}, \textit{Velar}, \textit{Glottal} \}
\end{aligned}$$

For example, if:

$$\begin{aligned}
\textit{Minus} &= \{ p, f, ð, s, t, k, h \} \\
\textit{Alveolar} &= \{ t, d, n, s, z \}
\end{aligned}$$

then

$$\begin{bmatrix} Voice & Minus \\ Place & Alveolar \end{bmatrix}$$

is a feature specification denoting $\{t, s\}$.