Statement Logic

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Logic

The program of logic

- The notion **valid argument**
- Analysis of **analytic truth**
- The notion **logical form**

The entailment relation $\Rightarrow$

- The entailment relation is a relation between between sentences (statements), or between sets of sentences (statements)
- $\{$ John is Albanian, Mary is Swedish $\} \Rightarrow \{$ John is Albanian and Mary is Swedish $\}$

  (1) If ’John is Albanian’ is true and ’Mary is Swedish’ is true, then ’John is Albanian and Mary is Swedish’ is true.
- $\{$ John is Albanian and Mary is Swedish $\} \Rightarrow \{$ John is Albanian, Mary is Swedish $\}$

  (2) If ’John is Albanian and Mary is Swedish’ is true then ’John is Albanian’ is true and ’Mary is Swedish’ is true.
- $p \Rightarrow q$

  If sentence ’p’ is true, then sentence ’q’ MUST be true.

Valid argument

- The entailment relation characterizes **valid arguments**
- A valid argument is an argument which is true **if the premises are true**
- Disjunctive Syllogism (an argument form)

  Either Mary left or John was unhappy.

  Mary did not leave.

  John was unhappy

- { Either Mary left or John was unhappy, Mary did not leave } $\Rightarrow$ { John was unhappy }

Interdisciplinary appeal

- The entailment relation is of interest to semanticists and to logicians.
- The entailment relation tells us something about meaning.
- The entailment relation plays a role in determining what arguments are valid.

Analytic truth

Mary is a tall basketball player $\Rightarrow$ Mary is a basketball player

If sentence 'Mary is a tall basketball player' is true, then the sentence 'Mary is a basketball player' MUST be true.

True in virtue of what?

True in virtue of the meanings of the sentences

Logical form

- Disjunctive Syllogism

  Either Mary left or John was unhappy.

  Mary did not leave.

  John was unhappy

- Abstract away from the parts of the meaning that are irrelevant
- 'or' ($\vee$) and 'not'($\neg$) are relevant
- $P \vee Q$

  $\neg Q$

  P

The big Lebowsky

Logicians don't care about whether a conclusion is true, only whether it preserves the truth of the premises. A valid argument:

(3)

All politicians always tell the truth.	Premise 1
Arnold Schwarzenegger is a politician.	Premise 2
Arnold Schwarzenegger always tells the truth.	Conclusion

(4)

All P are Q	Premise 1
x is P.	Premise 2
X is Q.	Conclusion

Tautology

A special case of a valid argument is a **tautology**. A tautology is a sentence that has to be true.

In other words, it's a one-sentence valid argument.

Either it's raining or it's not raining.

Summary

- **Valid argument:** an argument whose conclusion is true whenever the premises are true.
- **Entailment relation** $\Rightarrow$: Captures the relation that holds between sentences in a valid argument.
- **Analytic truth**: $\Rightarrow$ holds between sentences in virtue of their meaning and is of interest to both logicians and semanticists.
- We are interested in characterizing the **form** (or logical form) of valid arguments, not just listing arguments

Truth-Functional Connectives

A logical language

We will capture a certain kind of simple valid argument by studying a logical language called **statement logic** which attempts to capture the logical form of arguments whose validity depends on certain words that connect senetnces (**sentential connectives**):

...and...	$\wedge$
...or...	$\vee$
if...then	$\rightarrow$
...if and only if...	$\leftrightarrow$
not ...	$\neg$

Elements of statement logic

- A(n infinite) set of sentential variables (the arguments we are studying are all valid independently of what the sentences actually are):

$$p, q, r, s, t, u, v, \ldots$$

- The connectives:

$$\wedge, \vee, \rightarrow, \leftrightarrow, \neg$$

Syntax of statement logic

1. If α is a statement and β is a statement, then $\alpha \wedge \beta$ is a statement.
2. If α is a statement and β is a statement, then $\alpha \vee \beta$ is a statement.
3. If α is a statement and β is a statement, then $\alpha \rightarrow \beta$ is a statement.
4. If α is a statement and β is a statement, then $\alpha \leftrightarrow \beta$ is a statement.
5. If α is a statement, then $\neg\alpha$ is a statement.

Conjunction

Name	Form	Example
Conjunction and	$P \wedge Q$	Apples are sweet and radishes are bitter.

Truth Table		
P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Name	Form	Example
Disjunction or	$P \vee Q$	Apples are sweet or radishes are bitter.

Truth Table		
P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Material Implication

Name	**Form**	**Example**
Implication If... then, implies	$P \rightarrow Q$	If apples are sweet then radishes are bitter.

Truth Table		
P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

Name	Form	Example
Biconditional	$P \leftrightarrow Q$	Apples are sweet if and only if radishes are bitter.
...if and only if...		

Truth Table		
P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

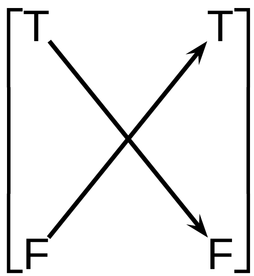
Negation

Name	Form	Example
Negation not	$\neg P$	Apples are not sweet.

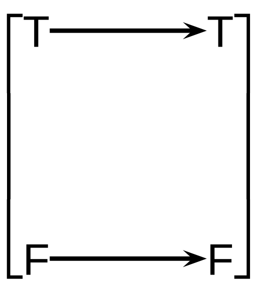
Truth Table	
P	$\neg P$
T	F
F	T

Connectives are Truth Functions

- Negation is one-place function from truth values to truth values.



- Other 1 place truth functions



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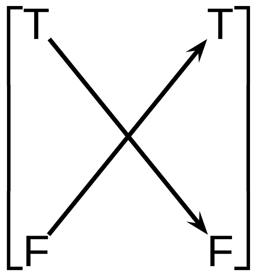

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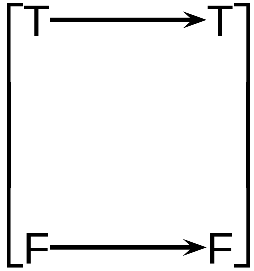
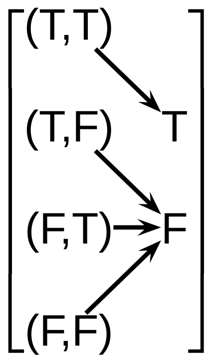


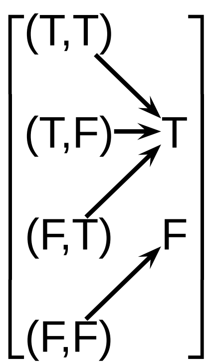
Corresponds to the English construction *It is true that....*

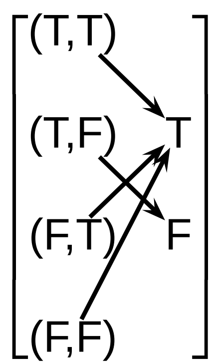
2 place Truth Functions

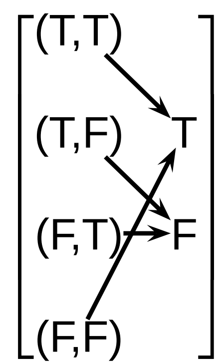
- Conjunction is a function from pairs of truth values to truth values.



- Other 2 place truth functions



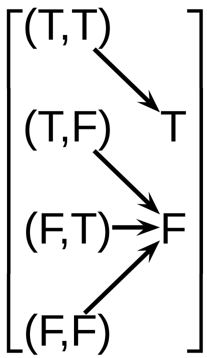




- How many distinct 2-place truth-functions are there?

2 place Truth Functions

- Conjunction is a function from pairs of truth values to truth values.



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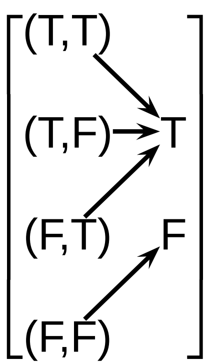
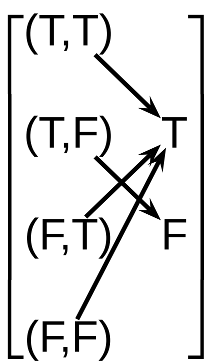
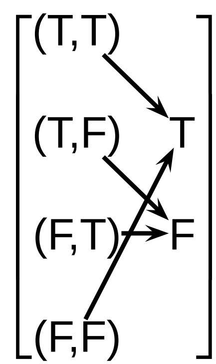


or, $\vee$





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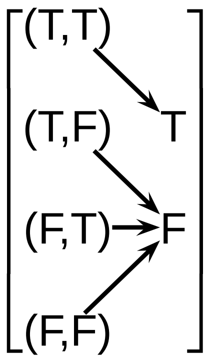

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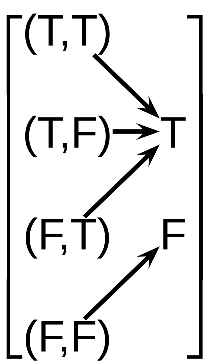

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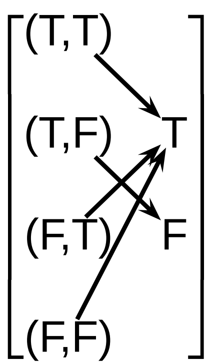
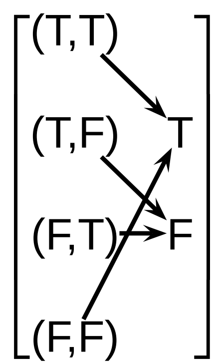


if..then..., $\rightarrow$



- How many distinct 2-place truth-functions are there?

2 place Truth Functions

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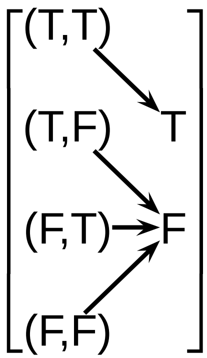

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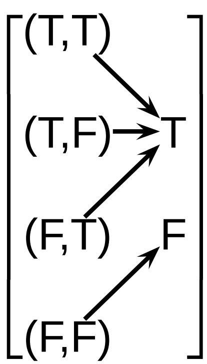
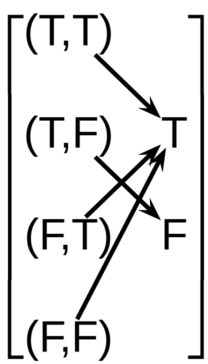
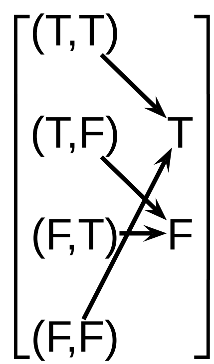


or, $\vee$



if..then..., $\rightarrow$

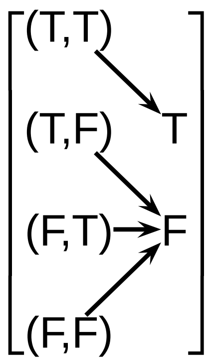


if and only if, $\leftrightarrow$

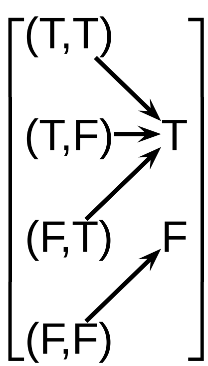
- How many distinct 2-place truth-functions are there?

2 place Truth Functions

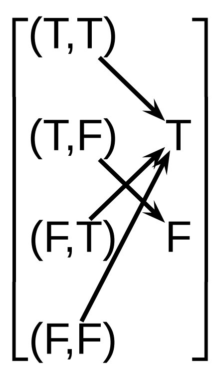
- Conjunction is a function from pairs of truth values to truth values.



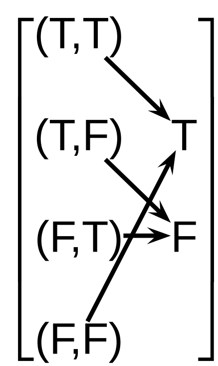
- Other 2 place truth functions



or, $\vee$



if..then..., $\rightarrow$



if and only if, $\leftrightarrow$

- How many distinct 2-place truth-functions are there? 16!

Laws for Truth Functional Connectives

Idempotent Laws

1.

(a) $(P \vee P) \Longleftrightarrow P$

(b) $(P \wedge P) \Longleftrightarrow P$

$P \vee P$ can always be substituted for P without changing the truth-value!

And vice versa!

Idempotent Laws

1.

(a) $(P \vee P) \Longleftrightarrow P$

(b) $(P \wedge P) \Longleftrightarrow P$

$P \vee P$ can always be substituted for P without changing the truth-value!

And vice versa!

Idempotent Laws proved!

Idea: Use truth-tables:

These two columns always have the same values!

The rows exhaust the possibilities.

P	$P \vee P$
T	T
F	F

Associative Laws

2.

(a) $((P \vee Q) \vee R) \Longleftrightarrow (P \vee (Q \vee R))$

(b) $((P \wedge Q) \wedge R) \Longleftrightarrow (P \wedge (Q \wedge R))$

Associativity for $\wedge$ proved!

These two columns always have the same values!

P	Q	R	$P \wedge Q$	$Q \wedge R$	$(P \wedge Q) \wedge R$	$P \wedge (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Commutative Laws

3.

(a) $(P \vee Q) \Longleftrightarrow (Q \vee P)$

(b) $(P \wedge Q) \Longleftrightarrow (Q \wedge P)$

Distributive Laws

4.

(a) $((P \vee Q) \wedge (P \vee R)) \Longleftrightarrow (P \vee (Q \wedge R))$

(b) $((P \wedge Q) \vee (P \wedge R)) \Longleftrightarrow (P \wedge (Q \vee R))$

A Distributive Law proved!

These two columns always have the same values!

P	Q	R	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Associativity for $\rightarrow$ disproved!

These two columns do NOT always have the same values!

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$(P \rightarrow Q) \rightarrow R$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	T
F	T	F	T	F	F	F
F	F	T	T	T	F	F
F	F	F	T	T	F	F

Identity Laws

5.

(a) $(P \vee F) \iff P$

(b) $(P \vee T) \iff T$

(c) $(P \wedge F) \iff F$

(d) $(P \wedge T) \iff P$

Complement Laws

6.

(a) $(P \vee \neg P) \Longleftrightarrow T$

(b) $\neg\neg P \Longleftrightarrow P$

(c) $(P \wedge \neg P) \Longleftrightarrow F$

De Morgan’s Laws

7.

(a) $\neg(P \vee Q) \iff (\neg Q \wedge \neg P)$

(b) $\neg(P \wedge Q) \iff (\neg Q \vee \neg P)$

Conditional Laws

8.

(a) $(P \to Q) \iff (\neg P \vee Q)$

(b) $(P \to Q) \iff (\neg Q \to \neg P)$

(c) $(P \to Q) \iff \neg(Q \wedge \neg P)$

Biconditional Laws

9.

(a) $(P \leftrightarrow Q) \iff ((P \rightarrow Q) \wedge (Q \rightarrow P))$

(c) $(P \leftrightarrow Q) \iff ((\neg Q \wedge \neg P) \wedge (P \wedge Q))$

Statement Logic Rules of Inference

Bridging Rules

Name and Abbr.	Form	Example
Modus Ponens	$P \rightarrow Q$	If Fred sang loud, Mary danced.
(M. T.)	P	Fred sang loud.
	Q	Mary danced.
Modus Tollens	$P \rightarrow Q$	If Fred sang loud, Mary danced.
(M. T.)	$\neg Q$	Mary didn’t dance.
	$\neg P$	Fred didn’t sing loud.

Syllogism Rules

Name and Abbr.	Form	Example
Hypothetical	$P \rightarrow Q$	If 3 is bigger than 2 then 3 is bigger than 1.
Syllogism	$Q \rightarrow R$	If 3 is bigger than 1 then 3 is bigger than 0.
(H. S.)	$P \rightarrow R$	If 3 is bigger than 2, then 3 is bigger than 0.
Disjunctive	$P \vee Q$	Either the bathroom is here or the bathroom is upstairs
Syllogism	$\neg P$	The bathroom is not here.
(D. S.)	Q	The bathroom is upstairs.

Simplification Rules

Name and Abbr.	Form	Example
Simplification (Simp.)	$\dfrac{P \wedge Q}{P}$	Apples are sweet and radishes are bitter. / Apples are sweet.

Conjunction Rules

Name and Abbr.	Form	Example
Conjunction	P	Apples are sweet.
(Conj.)	Q	Radishes are bitter.
	$P \wedge Q$	Apples are sweet and radishes are bitter.

Addition Rules

Name and Abbr.	Form	Example
Addition (Add.)	$\dfrac{P}{P \vee Q}$	Apples are sweet. / Apples are sweet or radishes are bitter.