

Functions

1 Introducing Functions

1.1 Domains and Ranges revisited

We define domain and range of a relation as the *projection* of the relation onto the first and second arguments respectively.

$$\begin{aligned}\text{Dom}(\mathbf{R}) &= \{x \mid \exists y \langle x, y \rangle \in \mathbf{R}\} \\ \text{Ran}(\mathbf{R}) &= \{y \mid \exists x \langle x, y \rangle \in \mathbf{R}\}\end{aligned}$$

So consider the *buy* relation as a subset of $\text{People} \times \text{Goods}$. The domain is the set of people who love someone, the range the set of goods that are bought by someone.

What is the set of People called (the set from which we choose possible first members of the *buy* relation)? Nothing. The left field of the relation? No! No good name.

What is the set of Goods called (the set from which we choose possible second members of the *buy* relation)? The right field? No. Nothing. No good name,

But where there is no good name we may still have a good notation Notation for defining “left and right fields” of relations:

$$\mathbf{R} : \mathbf{S} \longrightarrow \mathbf{T}$$

We read this as “a relation from S to T”.

Example:

- (a) $\text{Fricative2Stop} : \text{Fricatives} \longrightarrow \text{Stops}$
- (a) $\text{successor} : \mathbb{N} \longrightarrow \mathbb{N}$

1.2 Basic Notation and Examples

Functions are a special case of relations.

To be a function a relation f from S to T must meet two conditions:

1. The domain of f must equal S. That is, f must pair each member of S with some member of T.

$$\forall x \in \mathbf{S} \exists y [f(x) = y]$$

2. f must pair each member with S with only one value in T.

$$\forall x \in S \forall y, z \in T [\text{If } f(x) = y \text{ and } f(x) = z \text{ then } y = z]$$

Thus the buying relation is a function if and only if every person buys something and each person buys only one thing.

Under the conventional interpretation the father-of relation if the child is in the first coordinate position and the parent is in second. Every person has a father and every person only has one. The brother-of relation is not a function from people to people on two counts. Not every person has a brother and a person may have more than one.

We use the terms domain and range as above, for the projections of the relation onto the first and second coordinates respectively.

The notation above can be used for any relation. The following alternative notation, which uses \mapsto , is limited to functions:

$$\begin{aligned} \text{successor} &: x \mapsto x + 1 \\ \text{squared} &: x \mapsto x^2 \\ f &: x \mapsto x^2 + 1 \end{aligned}$$

This is closely related to the following notation you may have seen in algebra class:

$$\begin{aligned} \text{successor}(x) &= x + 1 \\ \text{squared}(x) &= x^2 \\ f(x) &= x^2 + 1 \end{aligned}$$

Notice it only makes sense to say

$$f(x) = y$$

if there's one and only one y that stands in f relation to x . So this notation, too, is limited to functions.

Note that f can also be defined:

$$f = \text{successor} \circ \text{squared}$$

Composition works with relations as it does with functions.
Be careful with reading function composition:

$$f \circ g(x) = f(g(x))$$

The second function mentioned gets applied first. This is sometimes read as *f following g*. Try:

$$\text{successor} \circ \text{squared}(2) = ?$$

$$\text{squared} \circ \text{successor}(2) = ?$$

We can also have functions of more than one argument. An example is Addition or +:

$$+ : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

Addition (+) maps from a pair of natural numbers to a natural number. Notice it's a function. For each pair of numbers there is and only one sum. Ditto for multiplication:

$$\times : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

Functions versus relations

N-ary Relation	n	Functional?	Why
Fricative2Stop	2	no	Not defined for all fricatives
OralStop2NasalStop	2	yes	
successor	2	yes	
squared	2	yes	
same-place	2	no	one-to-many
same-voicing	2	no	one-to-many
+	3	?	?
×	3	?	?
=	2	?	?
Passivization	2	?	?
Dative	2	?	?

2 One-to-one ness and onto ness

Definition 2.1. *One-to-one function*

We say a function is one-to-one if and only if

$$\forall x, y \{ \text{If } f(x) = f(y) \text{ then } x = y \}$$

Example 2.1. *Successor function is 1-1*

The successor function of arithmetic is 1-1. Every integer is the successor of exactly one integer.

Example 2.2. *Squaring function is not 1-1*

The squaring function is not 1-1. Every number has exactly two square roots. For example the square roots of 4 are 2 and -2.

Thus the squaring function maps two different numbers, 2 and -2, onto the same result.

Definition 2.2. *Onto functions*

We say a function f from S to T is onto if and only if

$$\forall y \in T \exists x \{f(x) = y\}$$

That is f is onto if and only if its range is equal to T .

Example 2.3. *Add-one function is onto*

We define the add-one function, which is like the successor function of arithmetic, except that it is defined for all integers (Z).

$$\text{add-one} = \{\langle x, y \in Z \times Z \mid y = x + 1 \rangle\}$$

Every integer is one more than some other integer, so add-one is onto.

Example 2.4. *Doubling function is not onto*

We define the double function, which is like the successor function of arithmetic, except that it is defined for all integers (Z).

$$\text{double} = \{\langle x, y \in Z \times Z \mid y = 2 \times x \rangle\}$$

Not every integer is twice than some other integer. Consider odd integers. So double is NOT onto.

3 Linguistic Rules as Functions

We define Passivization (kind of informally) as:

Passivization : Transitive Sentence \longrightarrow Sentence

Passivization : S \mapsto the passive version of S

Dative : Ditransitive Sentence \longrightarrow Sentence

Dative : S \mapsto the Dative moved version of S

Passivization:

1. **Transitive:** John ate the bagel.
2. **Mapped to:** The bagel was eaten by John.

Dative:

1. **Transitive:**The boy gave the book to the man
2. **Mapped to:** The boy gave the man the book..

Passivization(John ate the bagel) = The bagel was eaten by john

Dative(The boy gave the book to the man) = The boy gave the man the book.

Passive ◦ Dative(The boy gave the book to the man) = The man was given the book by the boy

Passive(??) = The book was given the man by the boy

4 Linguistic Features as functions

The sound-place relation is a linguistic feature that relates a sound to its place. It is a function. Every sound has a place, every sound has exactly one.

We can take any linguistic feature and call it a function from linguistic items to some convenient *feature value set*.

$$\text{sound-voicing} = \{ \langle x, y \rangle \in \text{sounds} \times \text{plusminus} \}$$

Here the set plusminus is the feature value set. Its members are:

$$\text{plusminus} = \{+, -\}$$

We use this value set for a special class of linguistic features called binary features. The features that are on or off, yes or no, like voicing, nasalization, continuousness (continuants can keep going, like fricatives, non continuants, like stops, can't).

5 Building equivalence relations from functions

For any function f from S to T we can do the following:

$$R_f = \{ \langle x, y \rangle \in S \times S \mid f(x) = f(y) \}$$

R is equivalence relation. Exercise: Show this.

Now if f is one-to-one, then R is just the identity relation on S .

But if not, then R is some other equivalence relation. Consider our same-place relation as an example, defined from the sound-place function.

Another example: Consider:

$$R_{\text{squared}} = \{\langle x, y \rangle \in \mathbb{Z} \times \mathbb{Z} \mid \text{squared}(x) = \text{squared}(y)\}$$

This partitions the integers into little classes of integers that have the same squares.

Examples?

Now consider any linguistic feature, such as voicing.

6 Characteristic functions

Any set S in a universe of discourse D has an associated function χ , called its *characteristic function*:

$$\chi_S : D \mapsto \text{truth-values}$$

$$\chi_S = \{\langle x, y \rangle \in D \times \text{truth-values} \mid \text{If } x \in S \text{ then } y = \text{true. Otherwise } y = \text{false} \}$$

The set of truth values is the set

$$\{\text{true, false}\}$$

Sometimes 1 and 0 are used instead.

This function gives the same information as the set. It gives true for everything in the set, false for everything not in the set. Characteristic functions should remind you of linguistic feature functions for binary features, the ones that use the feature set plusminus. They do the same work: classify something as yes or no for some property.

The characteristic function of the set of dogs is true of every dog and false of every non dog.

Problems:

1. Some people have proposed that the characteristic function of the set of dogs is the *meaning* of the English word *dog*.

$$\llbracket \text{dog} \rrbracket = \chi_{\text{dog}}$$

Discuss some problems for this idea as a general account for the semantics of nouns.

2. Is the following true?

$$\chi_{S \cap T} = \chi_S \cap \chi_T$$