



Equivalence and Classification

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Equivalence Relations

R: an Equivalence Relation

Definition 1. *A relation R in the set A is an equivalence relation in A if and only if*

- 1. R is reflexive on A ; and*
- 2. R is symmetric; and*
- 3. R is transitive.*

Example I: identity

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- Symmetry: Equality is symmetric. If $x = y$, then $y = x$.
- Transitivity: If $x = y$ and $y = z$, then $x = z$.

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- Transitivity: If $\langle x, y \rangle \in A \times A$, and $\langle y, z \rangle \in A \times A$, this shows $x, z \in A$. So $\langle x, z \rangle \in A \times A$.

Example III: SameVoicing, Same Location

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- Symmetry: If a sound x has the same voicing as a sound y , then y has the same voicing as x .
- Transitivity: Suppose x has the same voicing as y and y has the same voicing as z , then x has the same voicing as z .

Proof: Suppose x is voiced. Then if x stands in the same voicing relation to y , then y is voiced. And if y stands in the same voicing relation to z , then z is voiced. So x and z have the same voicing. The case of x being voiceless is completely parallel.

Difference is divisible by 3

Consider this R

$$R = \{ \langle a, b \rangle \mid a - b \text{ is evenly divisible by } 3 \}$$

A list of numbers a that stand in the relation R when $b = 7$:

a	1	4	7	10	13	16	...
$a - 7$	-6	-3	0	3	6	...	

Notice all the numbers in the top row stand in the relation R to each other, because they are all separated by multiples of 3.

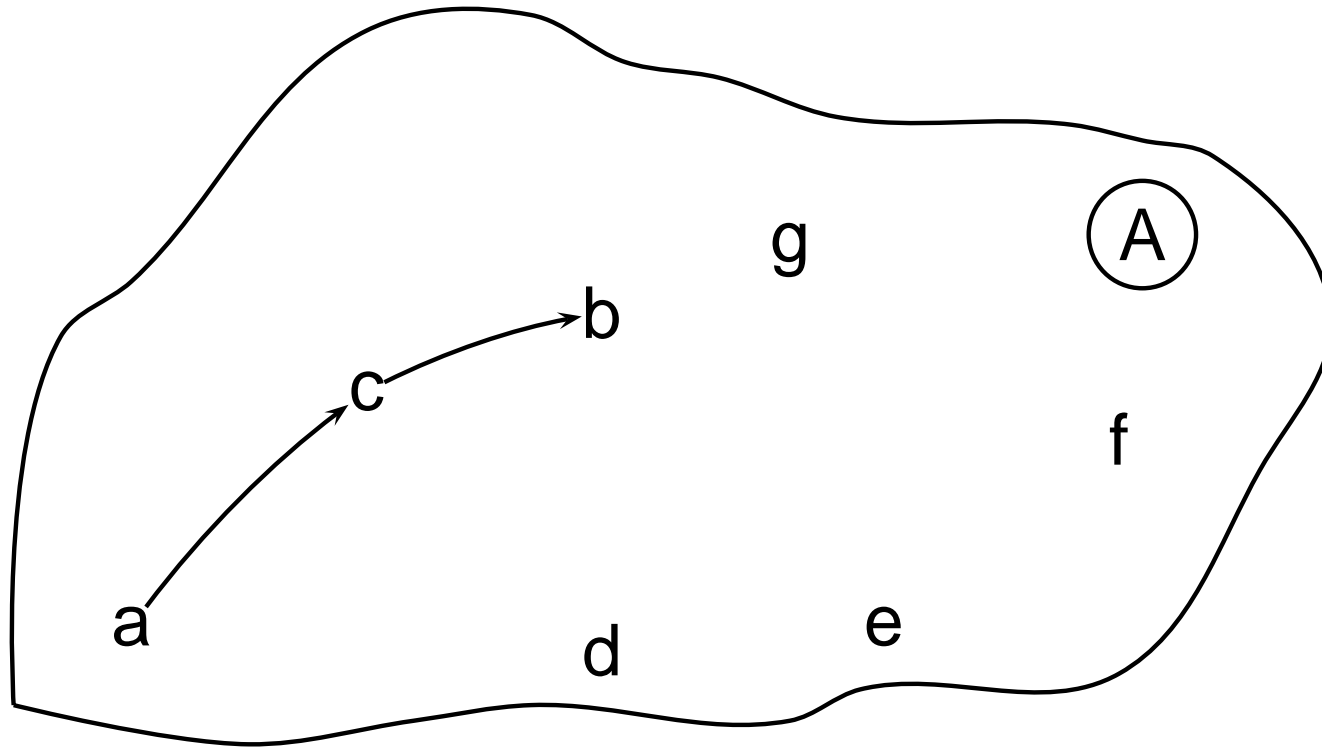
Claim: For any integer a , $a + m$ stands in the relation R to a , where m is a multiple of 3, and all the integers that stand in R to a stand in R to each other.

- Reflexivity: $x - x = 0$ and 0 is divisible by 3.
- Symmetry: If $x - y$ is divisible by 3, then $y - x$ is divisible by 3.
- Transitivity: Proof left to you.

Non-examples

- Sybling relation
- $\{\langle x, y \rangle \mid$
height difference of x and y is less than 5 inches}
- $\{\langle x, y \rangle \mid x \leq y\}$
- $\{\langle x, y \rangle \mid$
 x and y are people who eat at the same restaurant}

Equivalence relations: The intuition

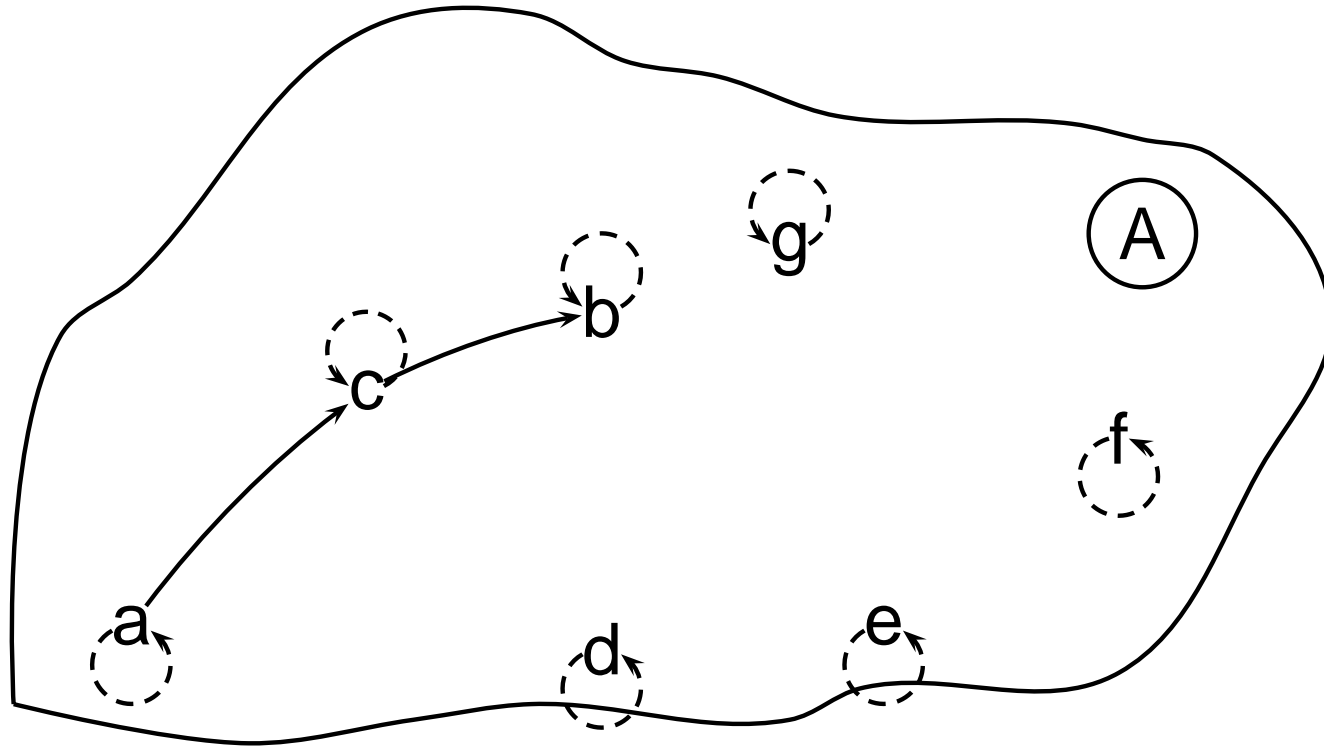


If R is an equivalence relation, then when the solid links exist, all links must exist.

Let $B = \{a, b, c, d, e, f\}$

Then $B \times B \subseteq R$

Equivalence relations: The intuition

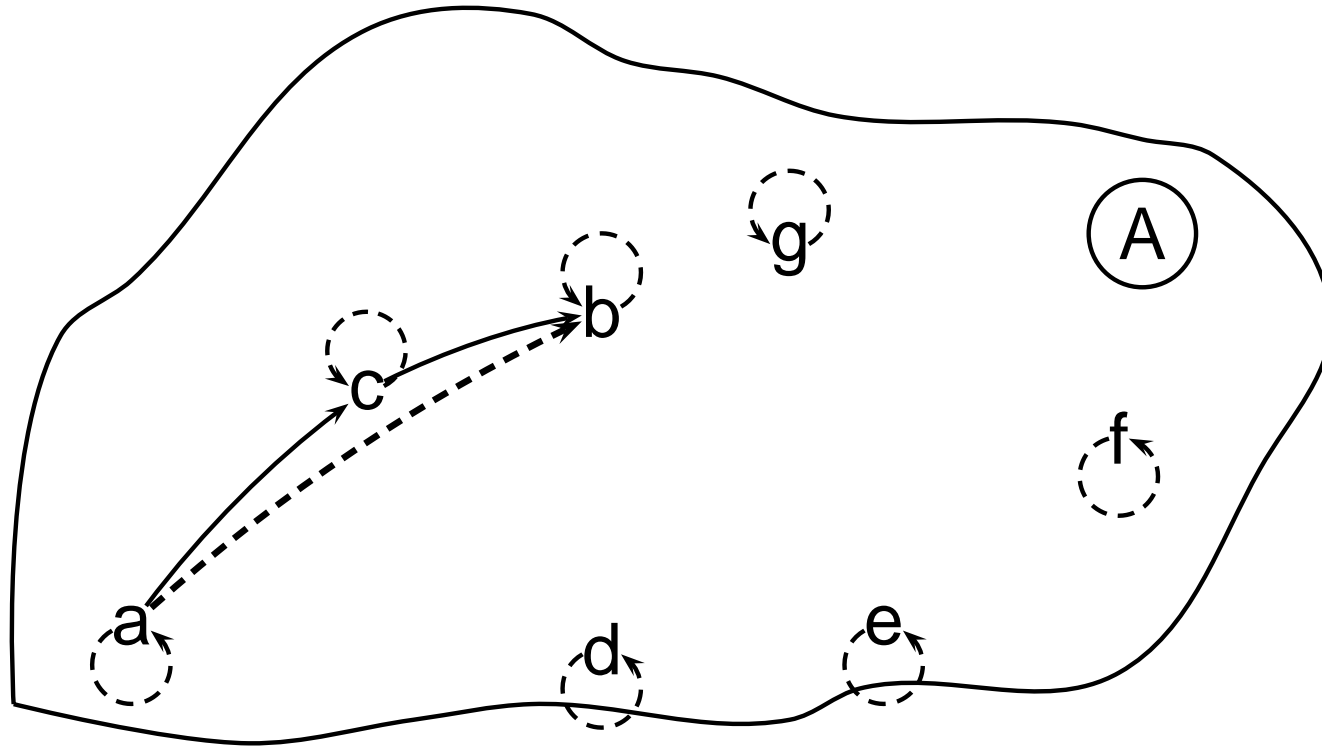


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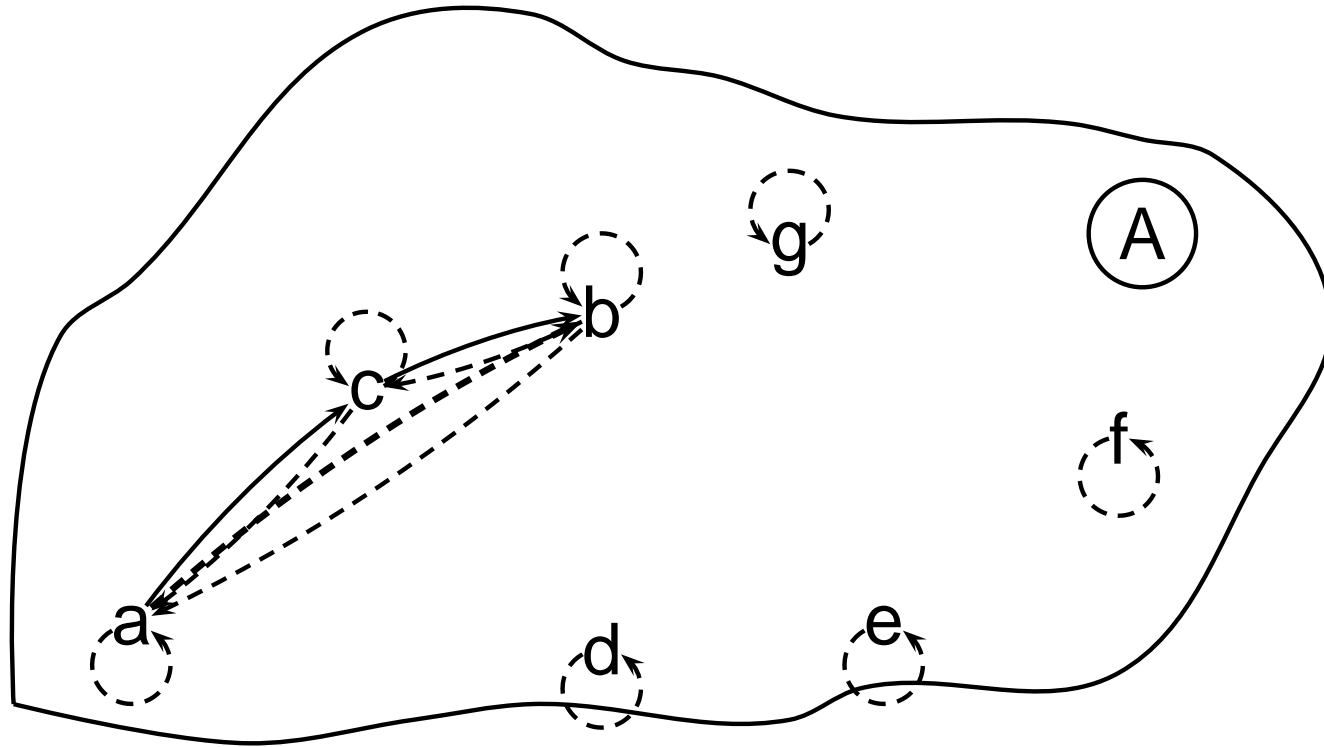


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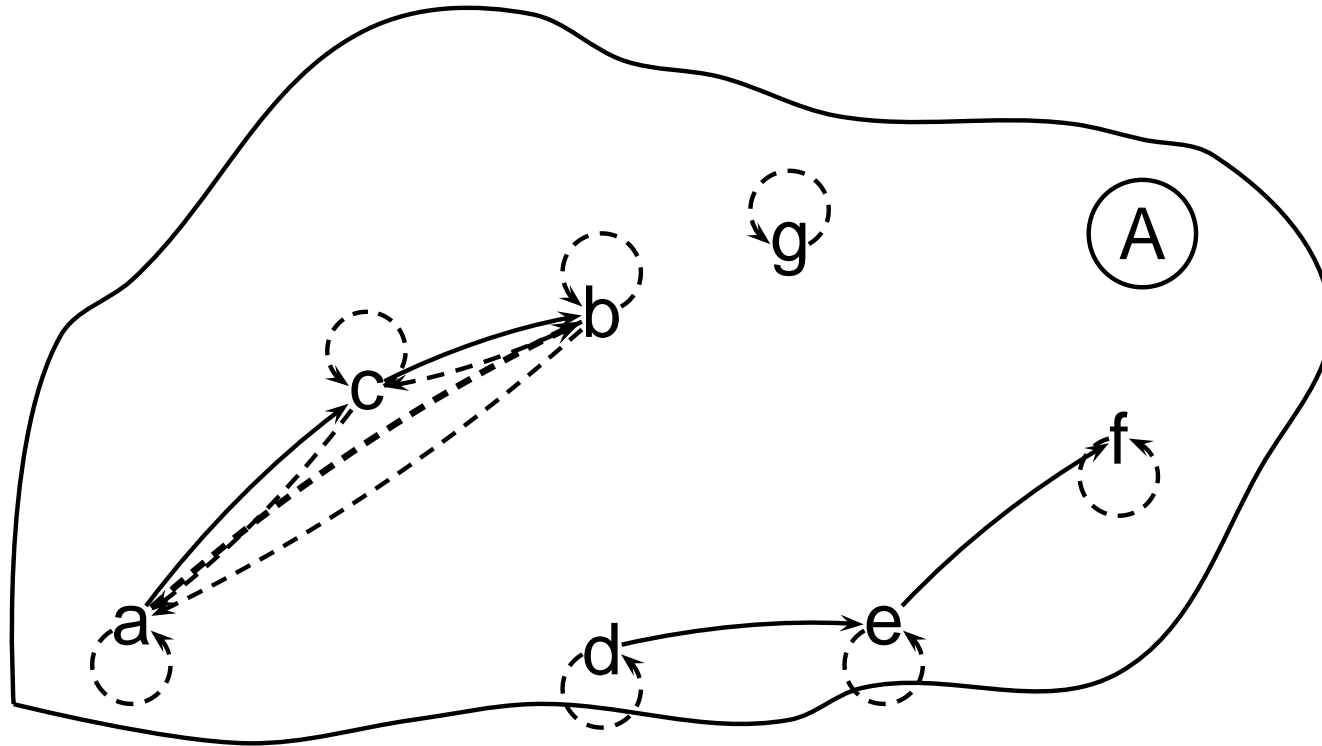


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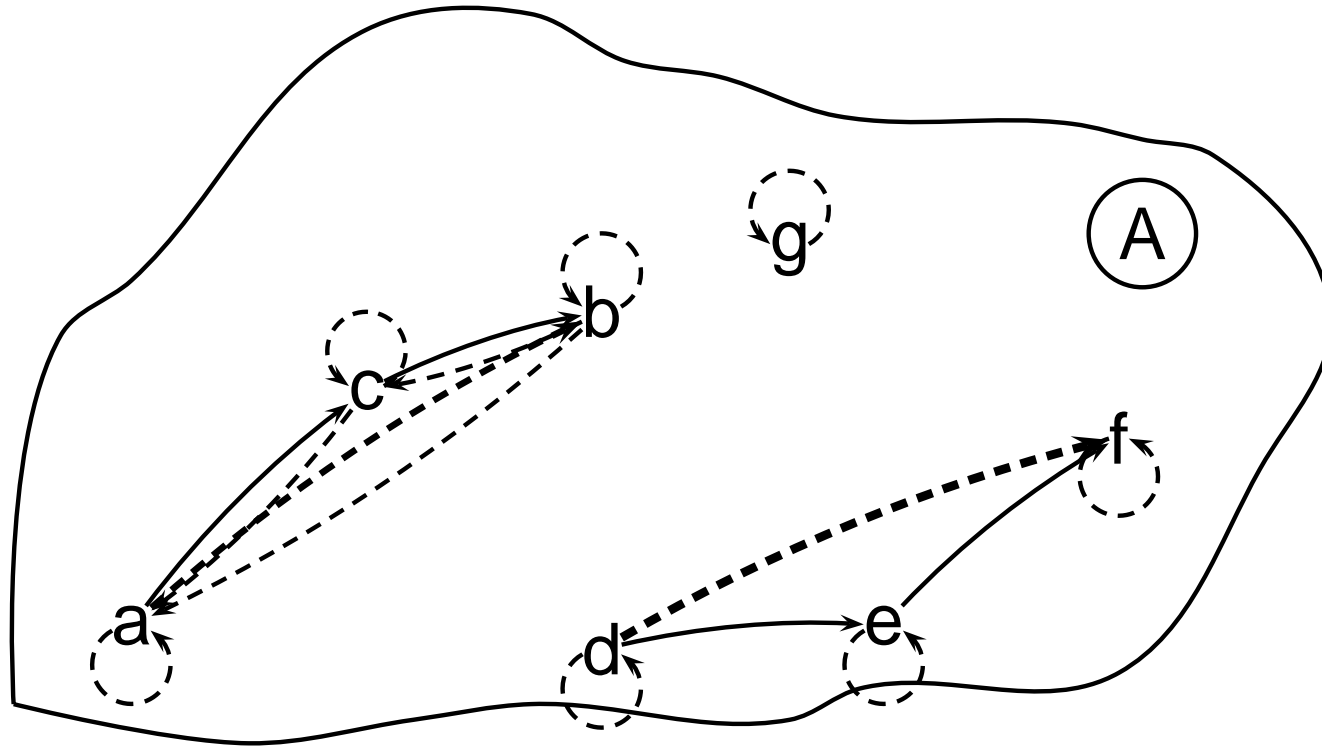


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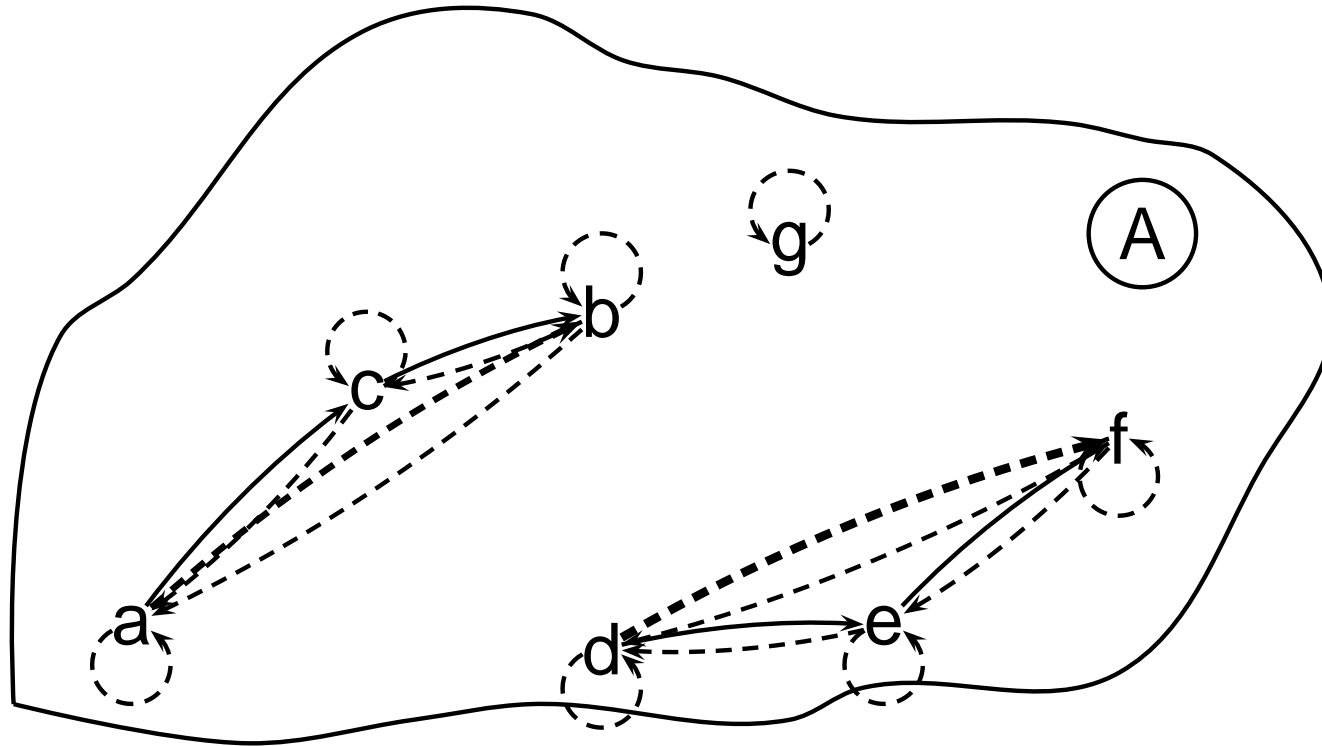


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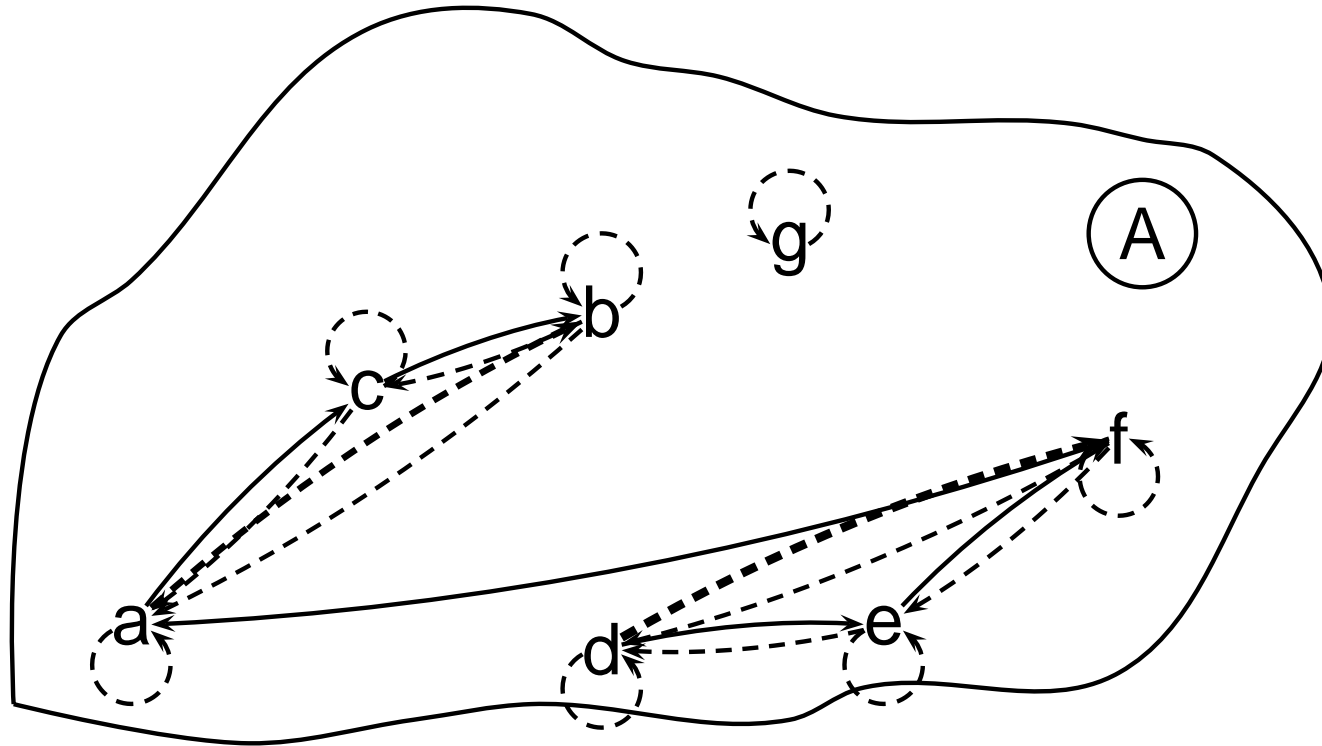


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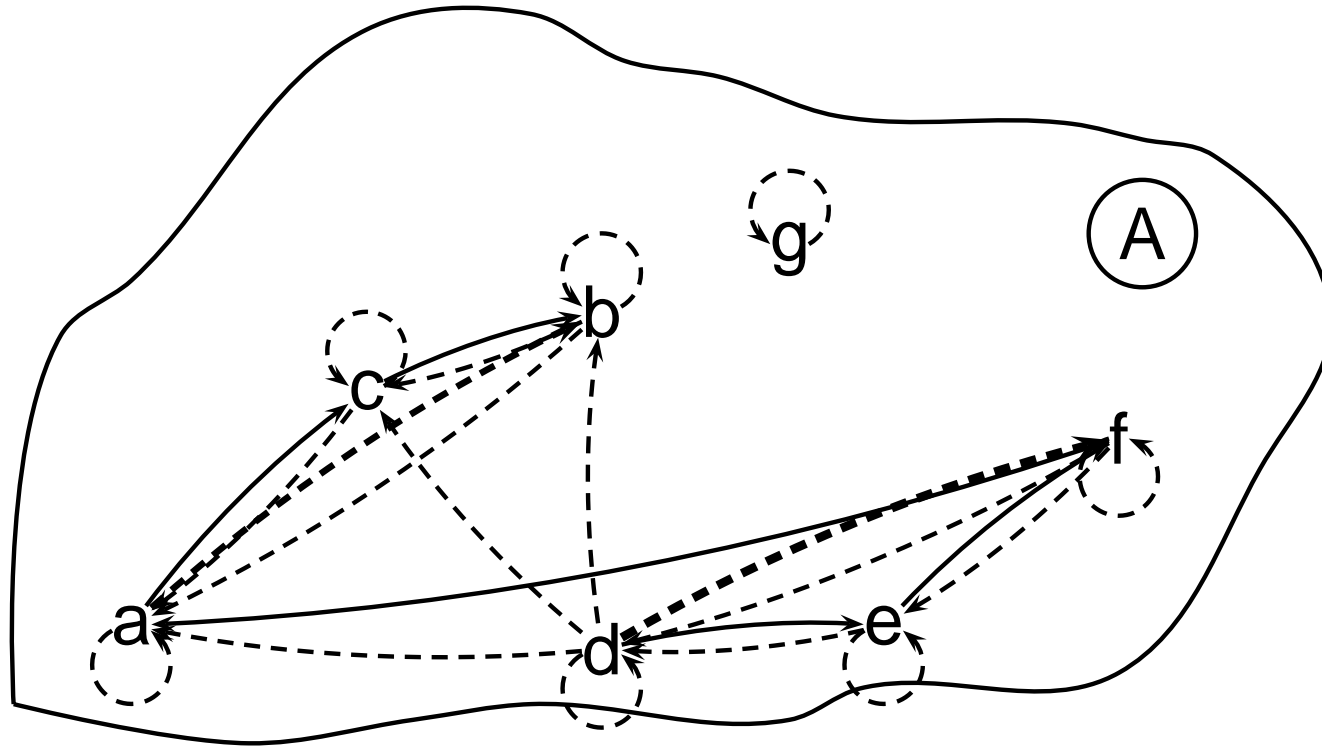


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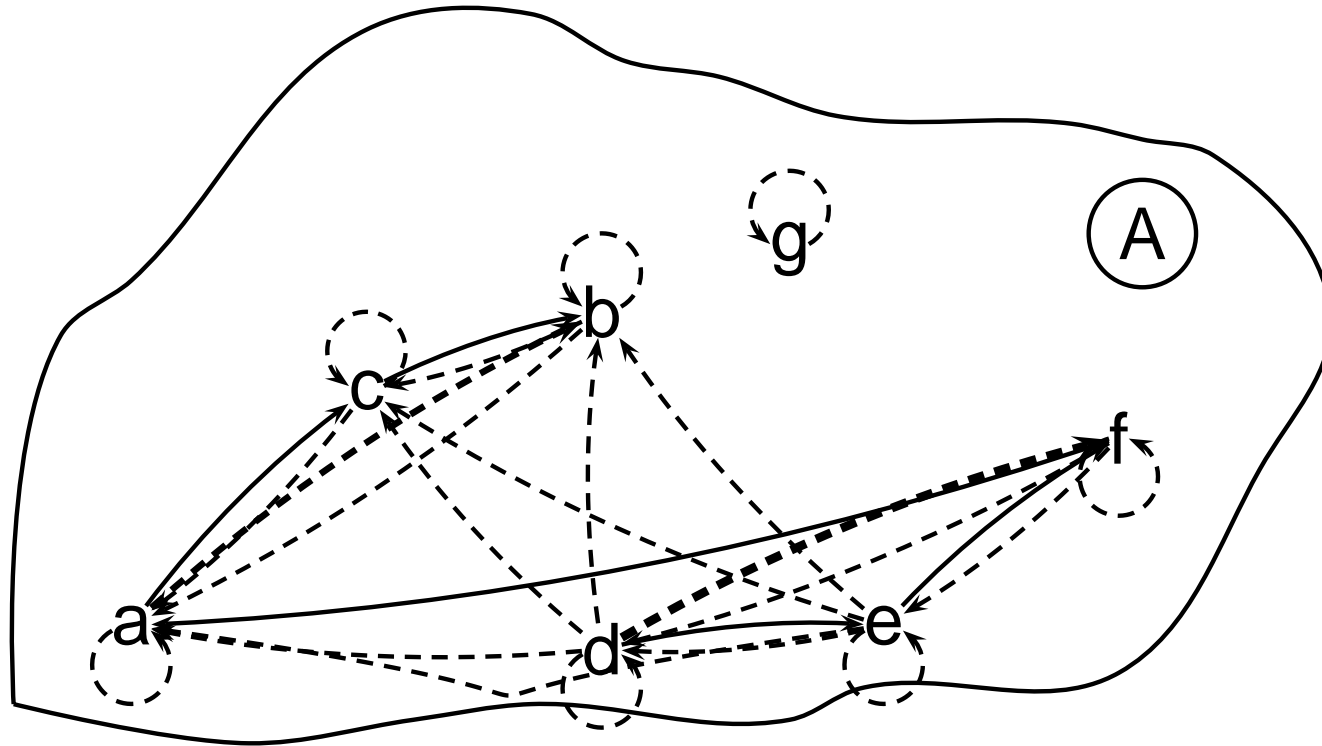


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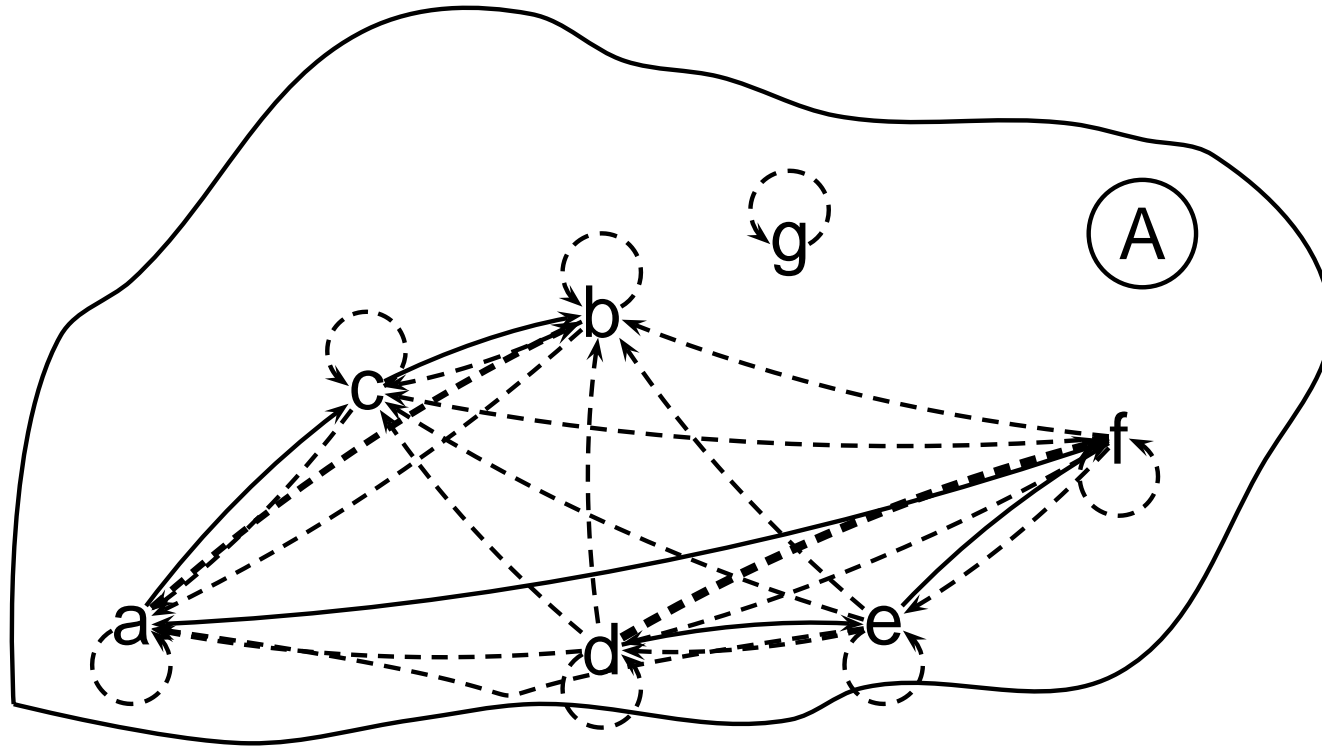


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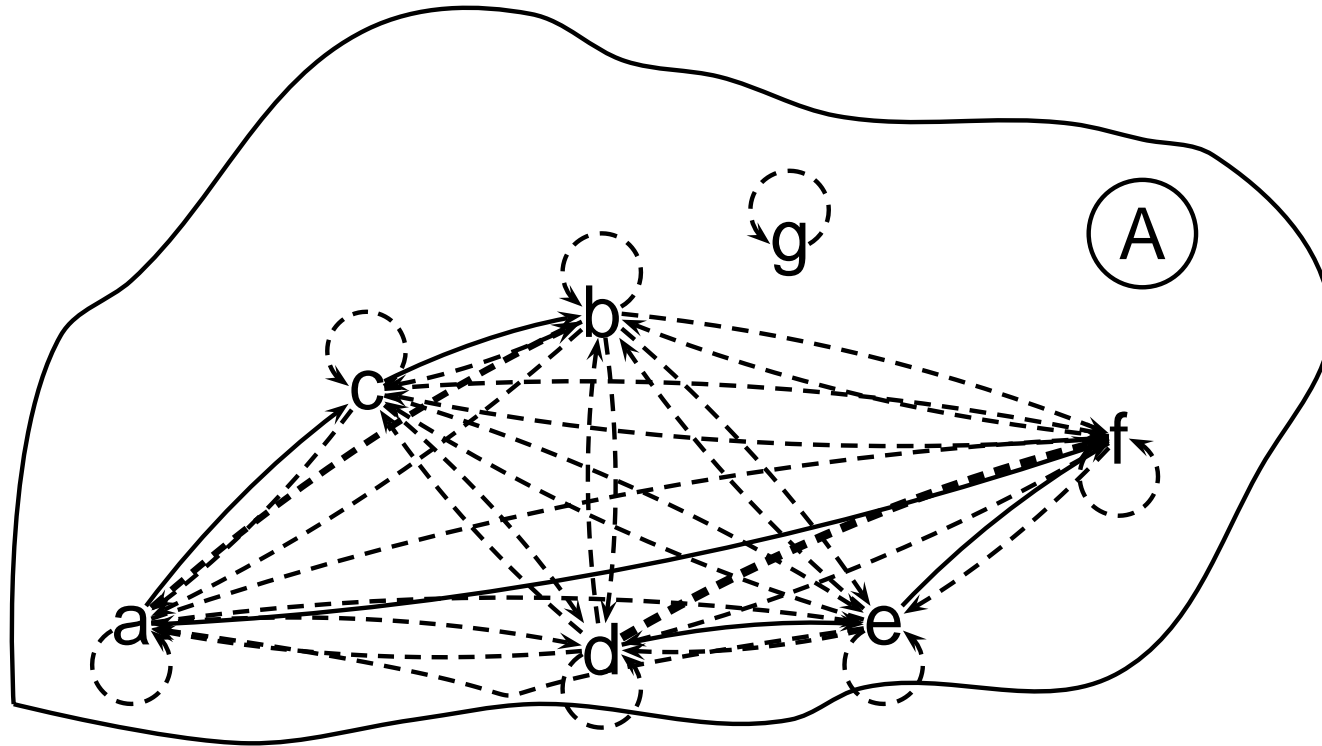


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Equivalence relations: Partitions

Definition 2. A partitioning of a set A is a set of disjoint sets p_1, p_2, \dots, p_n such that

$$A = p_1 \cup p_2 \cup p_3 \cup \dots \cup p_n$$

Definition 3. An equivalence relation R on A defines a partitioning of R . Each of the sets in the partition is called an equivalence class.

We define π_R , the partition induced by equivalence relation R , as follows.

$$\pi_R = \{S \mid \exists x \in A [S = \{y \mid \langle x, y \rangle \in R\}]\}$$

Partitions: The two extreme cases

- $|\pi| = |A|$. The identity relation on a set $A = \{a, b, c\}$ induces the partition:

$$\pi_I = \{ \{a\}, \{b\}, \{c\} \}$$

- $|\pi| = 1$. If $A = \{a, b, c\}$, then $A \times A$ induces the partition:

$$\pi_{A \times A} = \{ \{a, b, c\} \}$$

- Exercise: List the other possible partitions of A . Give the relations that induce them.

Difference is divisible by 3

The **difference is divisible by 3** relation partitions the set of integers into three sets we'll call S_0 , S_1 , and S_2 :

$$S_0 = \{0, 3, 6, 9, \dots\}$$

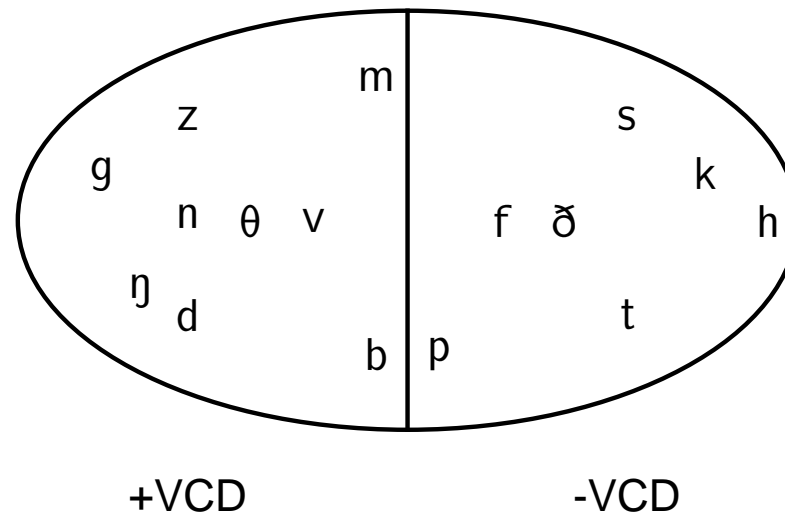
$$S_1 = \{1, 4, 7, 10, \dots\}$$

$$S_2 = \{2, 5, 8, 11, \dots\}$$

Note we have 3 disjoint **infinite** sets which unioned together give us the entire set of integers.

SameVoicing

We call the partition induced by the SameVoicing relation **Voice**



SamePlace

We will call the partition induced by the SamePlace relation **Place**:

p			t		k	
b			d		g	
m			n		ŋ	
	f					
	v					
		θ	s	ʃ		h
		ð	z	ʒ		

Definition 4. A **feature space** is a pair of a set A together with a set Π of partitions of A . A is called the domain of the feature space. Each member of Π is called a **feature**.

Example 1. Let us choose the set of obstruents as our domain, and the partitions induced by the SameVoice relation and the SamePlace relation as our features. Then

$$\text{PhonFeatures} = \langle \text{Obstruents}, \{ \text{Place}, \text{SameVoice} \} \rangle$$

is a feature space with *Obstruents* as its domain and

$$\{ \text{Place}, \text{Voice} \}$$

as its features.

Feature Specification

Definition 5. A **feature specification** ξ chosen from a feature space F is a set S such that each member of ξ is a **member** of one of the features of F .

If

$$\xi = \{ Val1, Val2 \} \quad \text{where } Val1 \in Feat1, Val2 \in Feat2$$

in feature space

$$F = \langle A, \{Feat1, Feat2\} \rangle,$$

then feature specification ξ can be written

$$\xi = \begin{bmatrix} Feat1 & Val1 \\ Feat2 & Val2 \end{bmatrix}$$

We call $\llbracket \xi \rrbracket$ the **denotation** of a feature specification, defined as the intersection of its partitions:

$$\llbracket \xi \rrbracket = Val1 \cap Val2$$

Feature Specs in PhonFeatures

Example 2. *Let:*

$$\begin{aligned} \textit{PhonFeatures} &= \langle \textit{Obstruents}, \{ \textit{Place}, \textit{Voice} \} \rangle \\ \textit{Voice} &= \{ \textit{Plus}, \textit{Minus} \} \\ \textit{Place} &= \{ \textit{Labial}, \textit{Labiovelar}, \textit{Interdental}, \textit{Alveolar} \\ &\quad \textit{Alveopalatal}, \textit{Velar}, \textit{Glottal} \} \end{aligned}$$

For example, if:

$$\begin{aligned} \textit{Minus} &= \{ p, f, \theta, s, t, k, h \} \\ \textit{Alveolar} &= \{ t, d, n, s, z \} \end{aligned}$$

then

$$\left[\begin{array}{cc} \textit{Voice} & \textit{Minus} \\ \textit{Place} & \textit{Alveolar} \end{array} \right]$$

is a feature specification denoting {t, s}.