Symmetries of the Square

Jean Mark Gawron
Codes and Language

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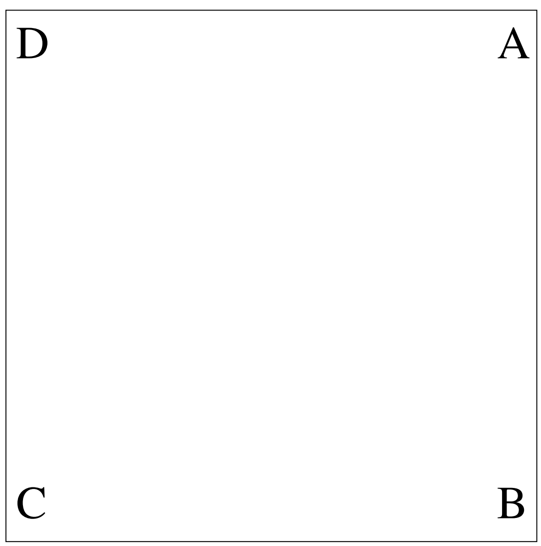
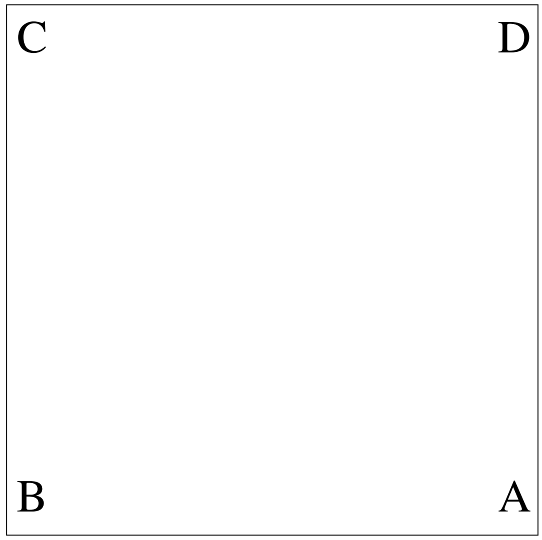
1 Basic Elements

A square:

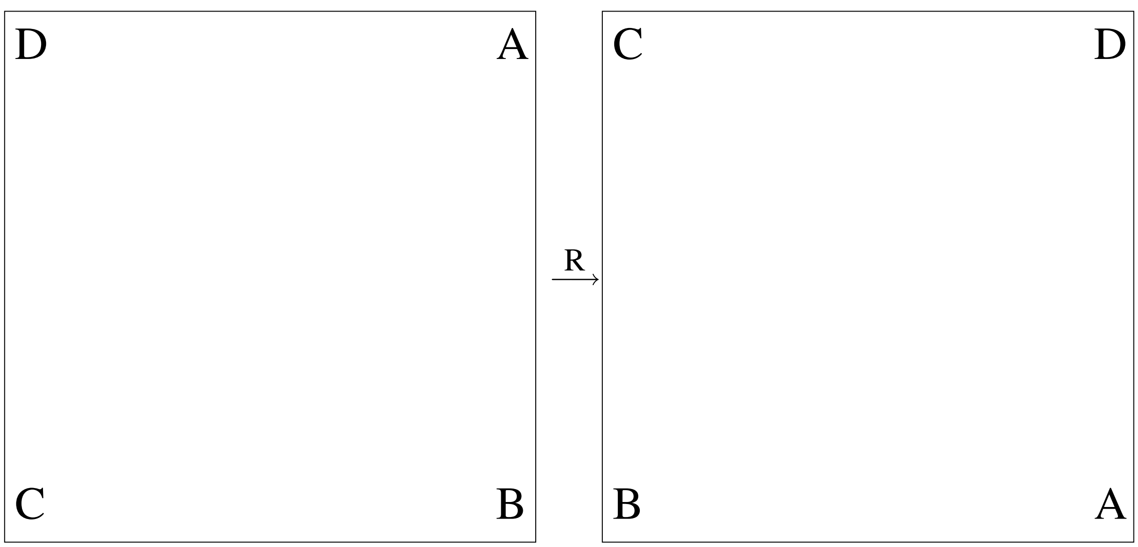


1.1 Rotations

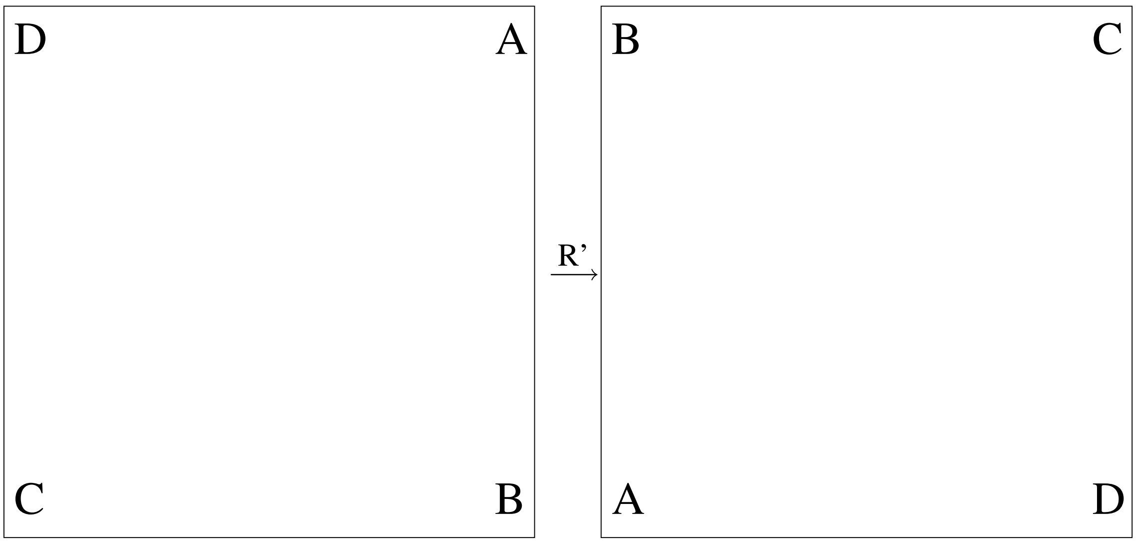
Rotate it $90°$ clockwise:



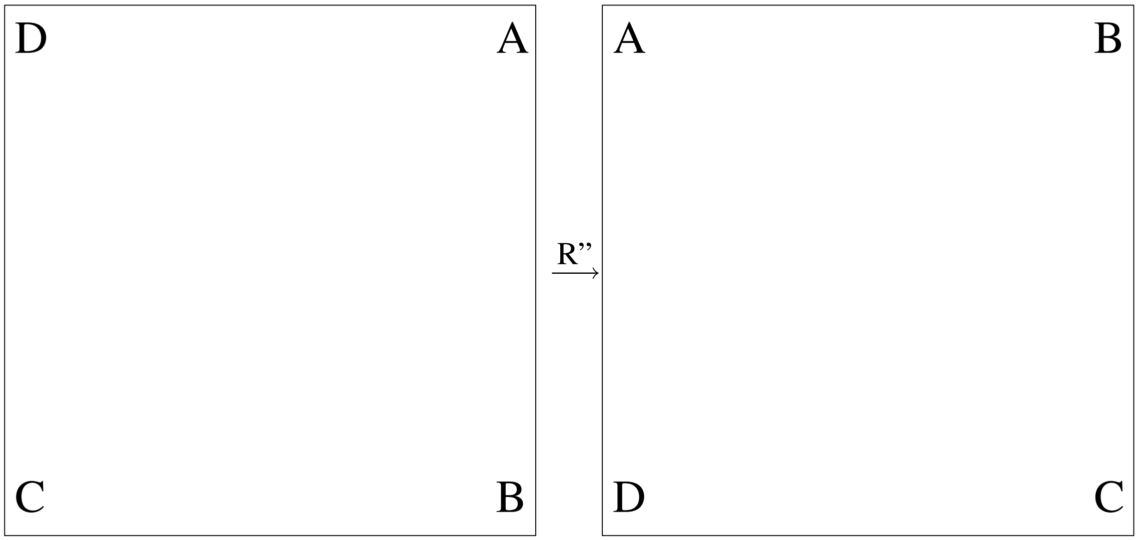
We call this operation R. So:



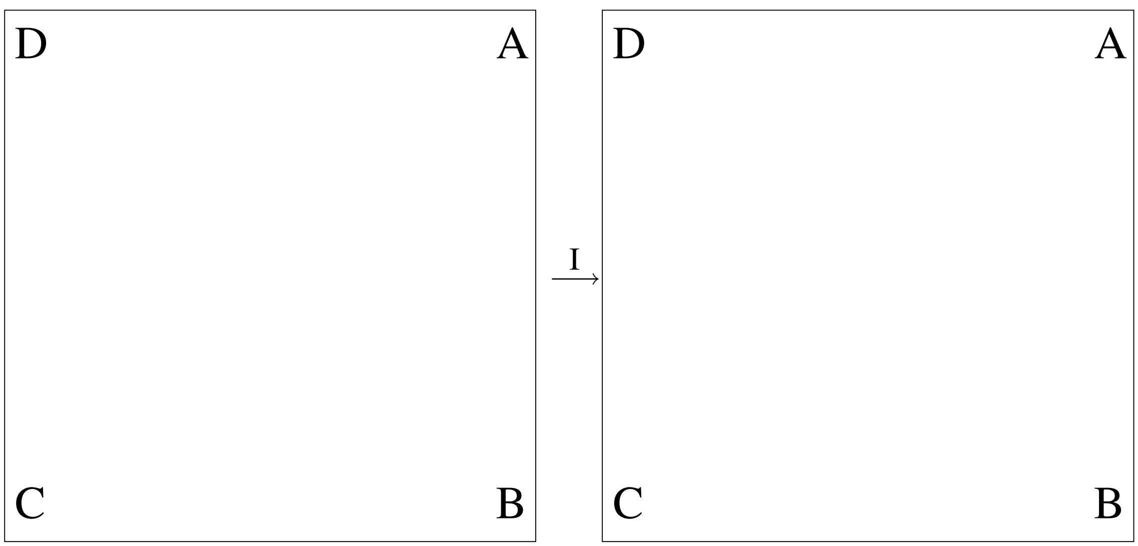
We call rotating 180° clockwise R'. So:



We call rotating $270°$ R”. So:



We call rotating $360°$ I (for identity). So:

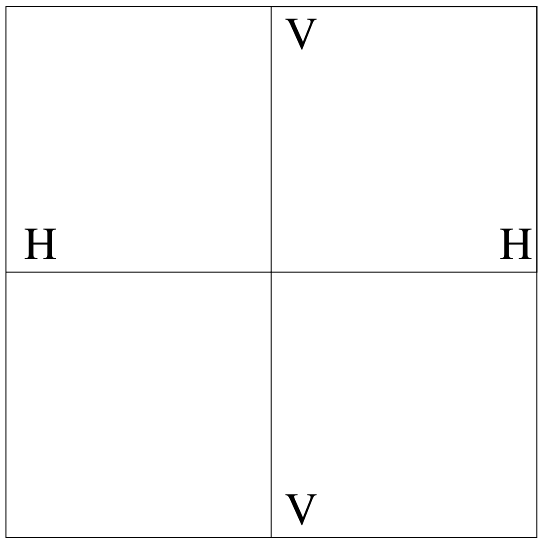


That completes the rotations.

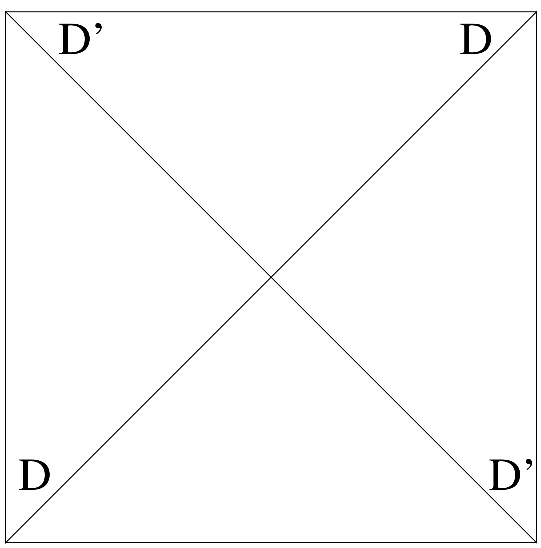
1.2 Reflections

Now consider *reflections around some axis of the square*

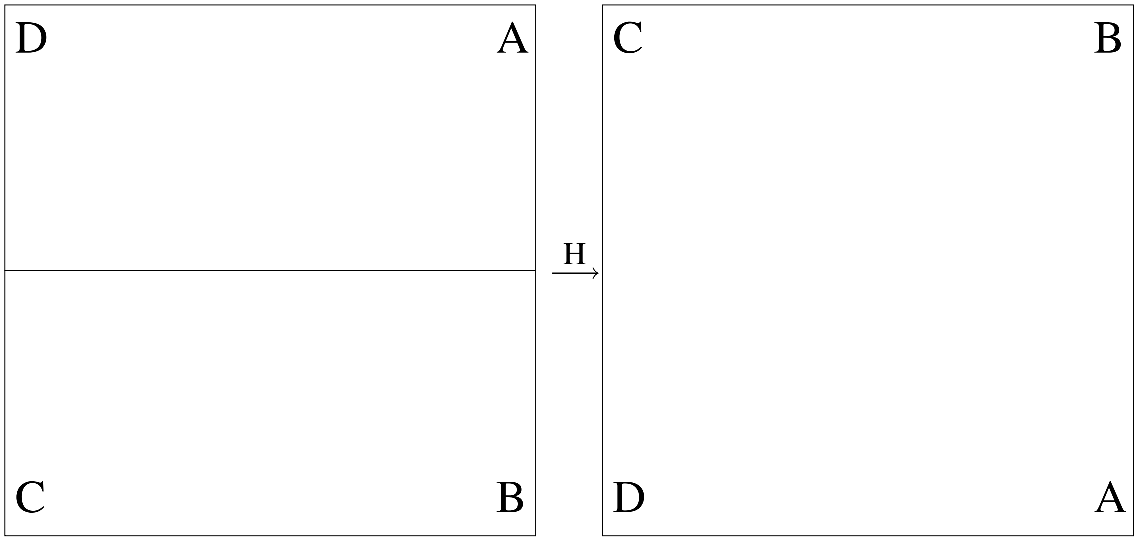
There are 4 axes in question. First H and V:

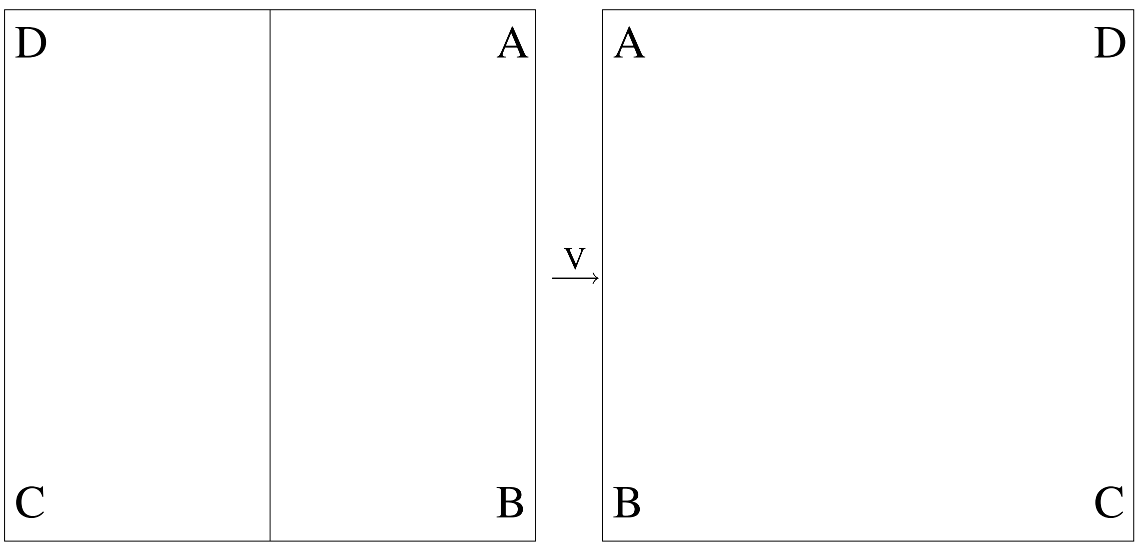


Then D ($x = y$) and D' ($x = -y$):

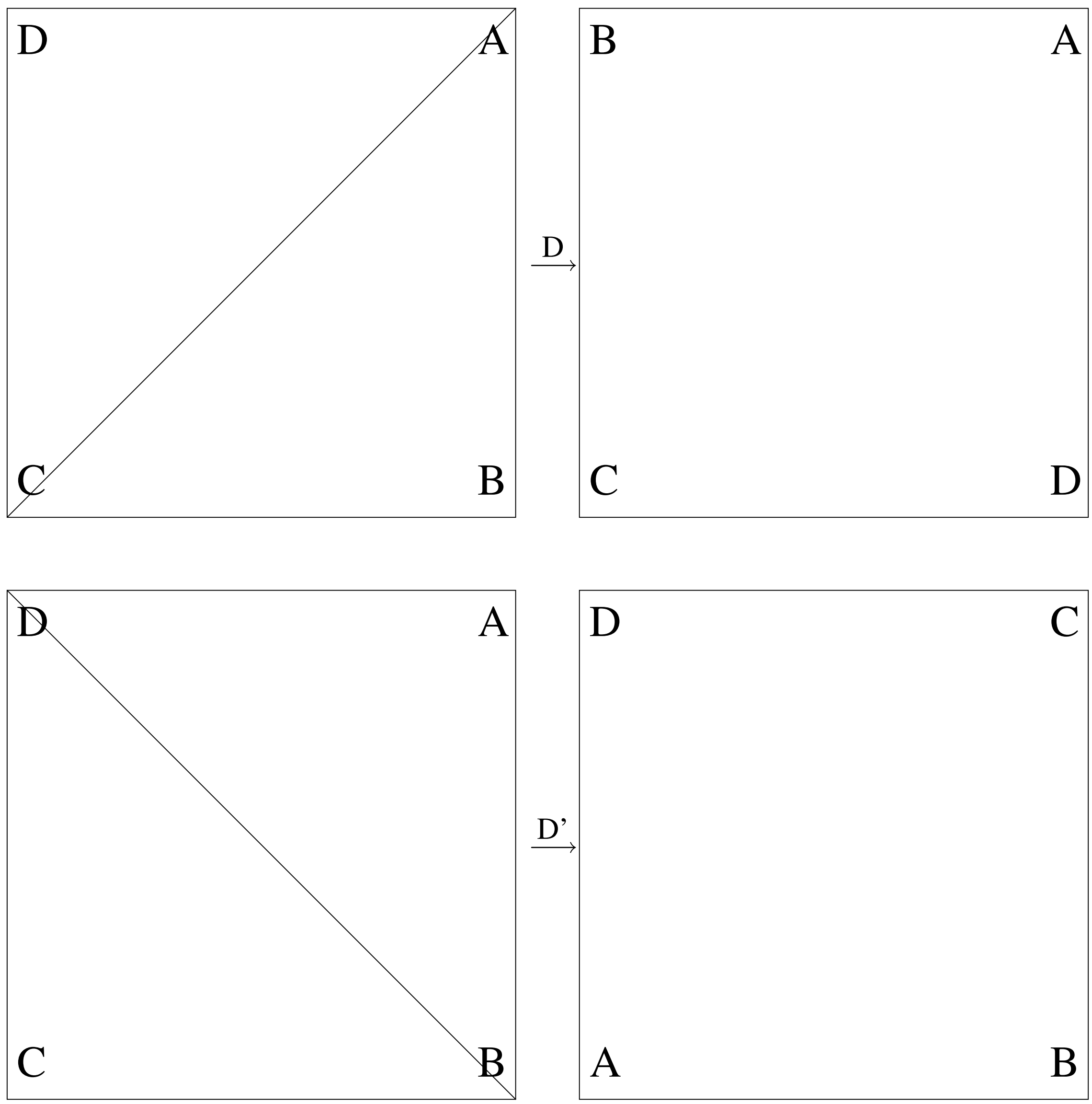


We illustrate H and V:



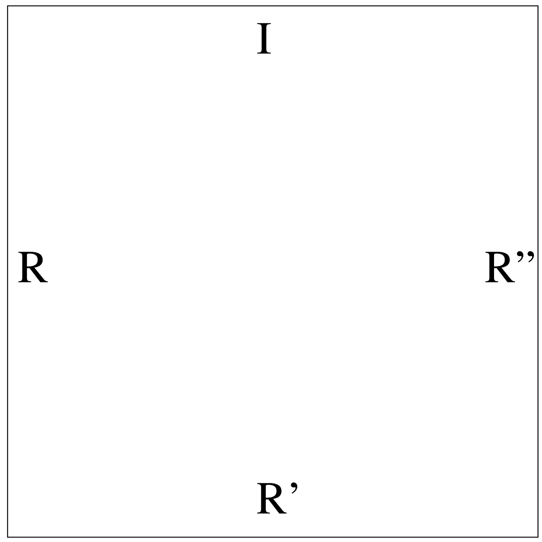


We illustrate D and D':



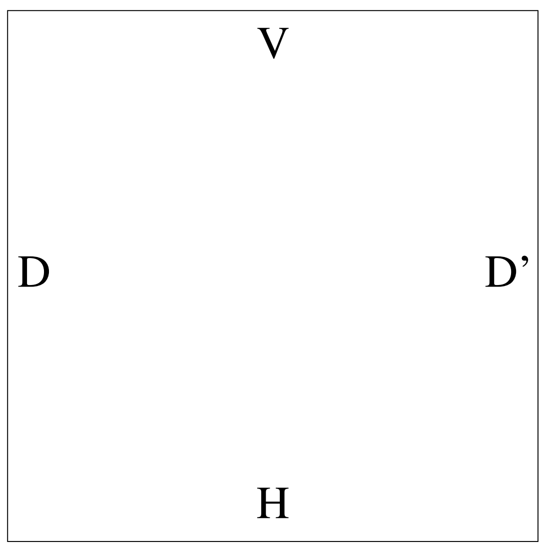
1.3 Summarizing Operations

The following square summarizes the rotations:



If you start with I on top, then performing an operation puts the side labeled with the operation’s name on top. For example, performing operation R puts R on top.

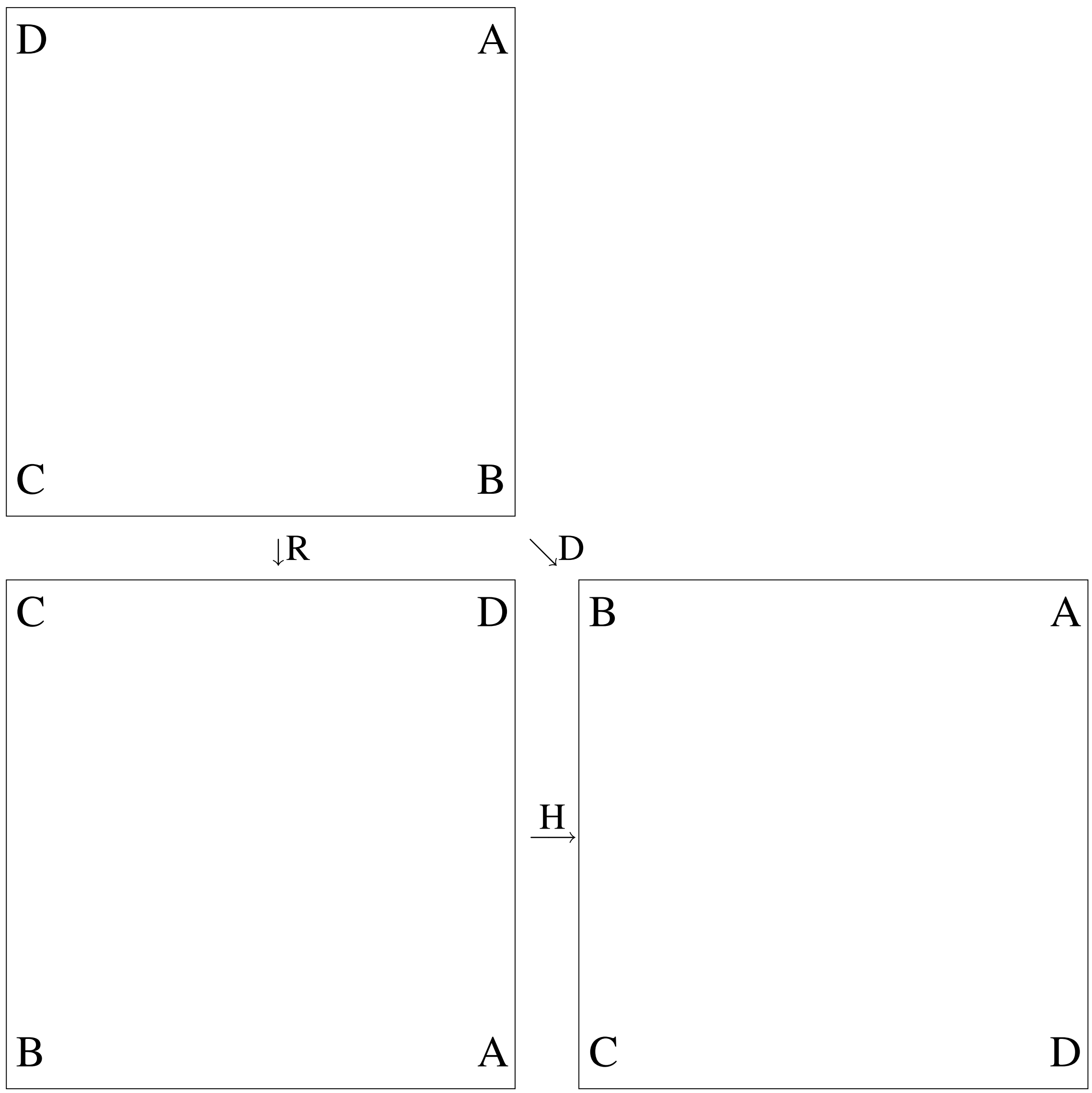
The following square summarizes the reflections:



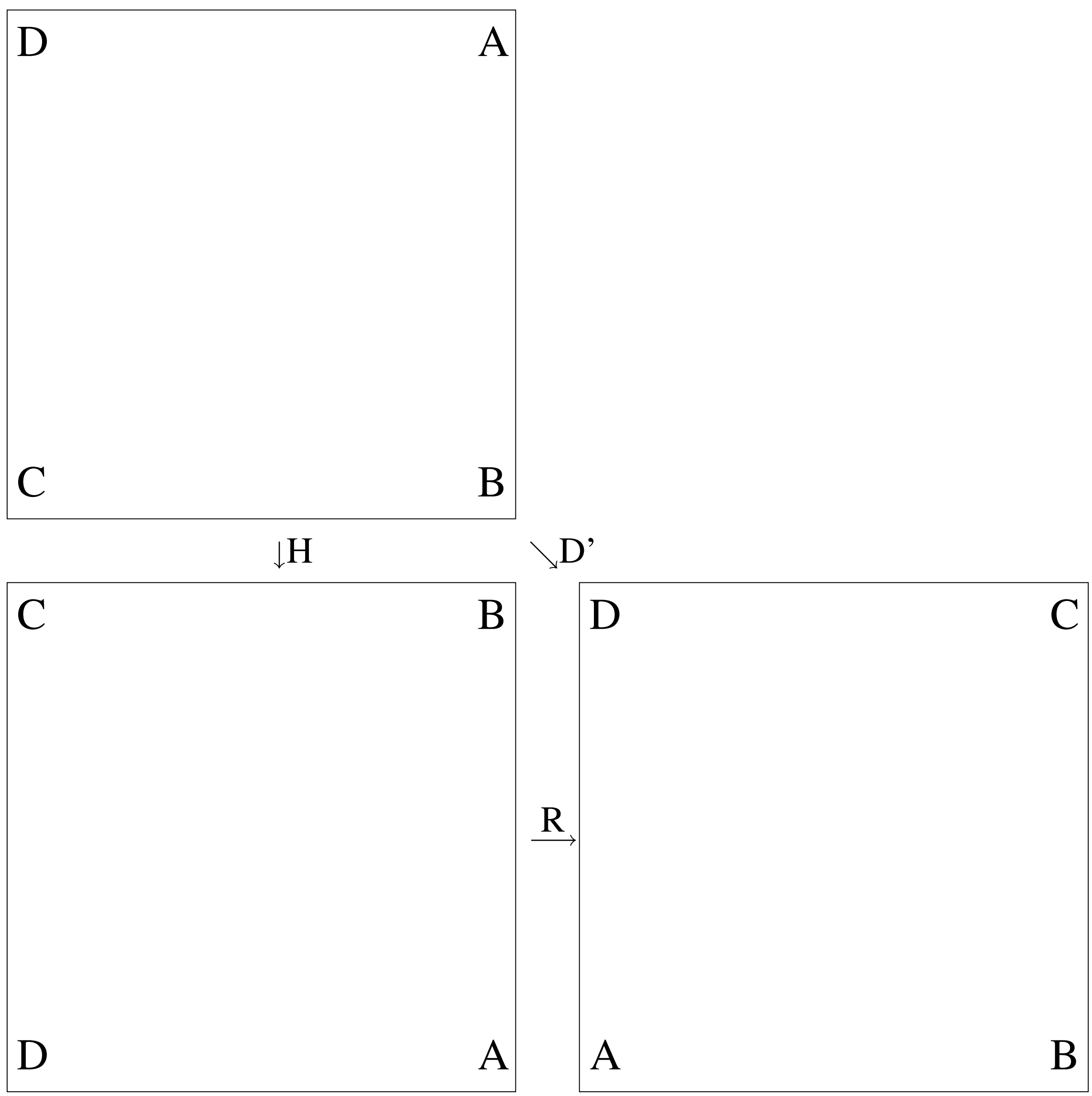
Performing a reflection always flips the square putting back to front. So put this digram on the back of the square, with V aligned with I. then if you start with the I side facing you, with I on top, then performing a particular reflection always ends up with side labeled with that reflection on top.

1.4 Composition

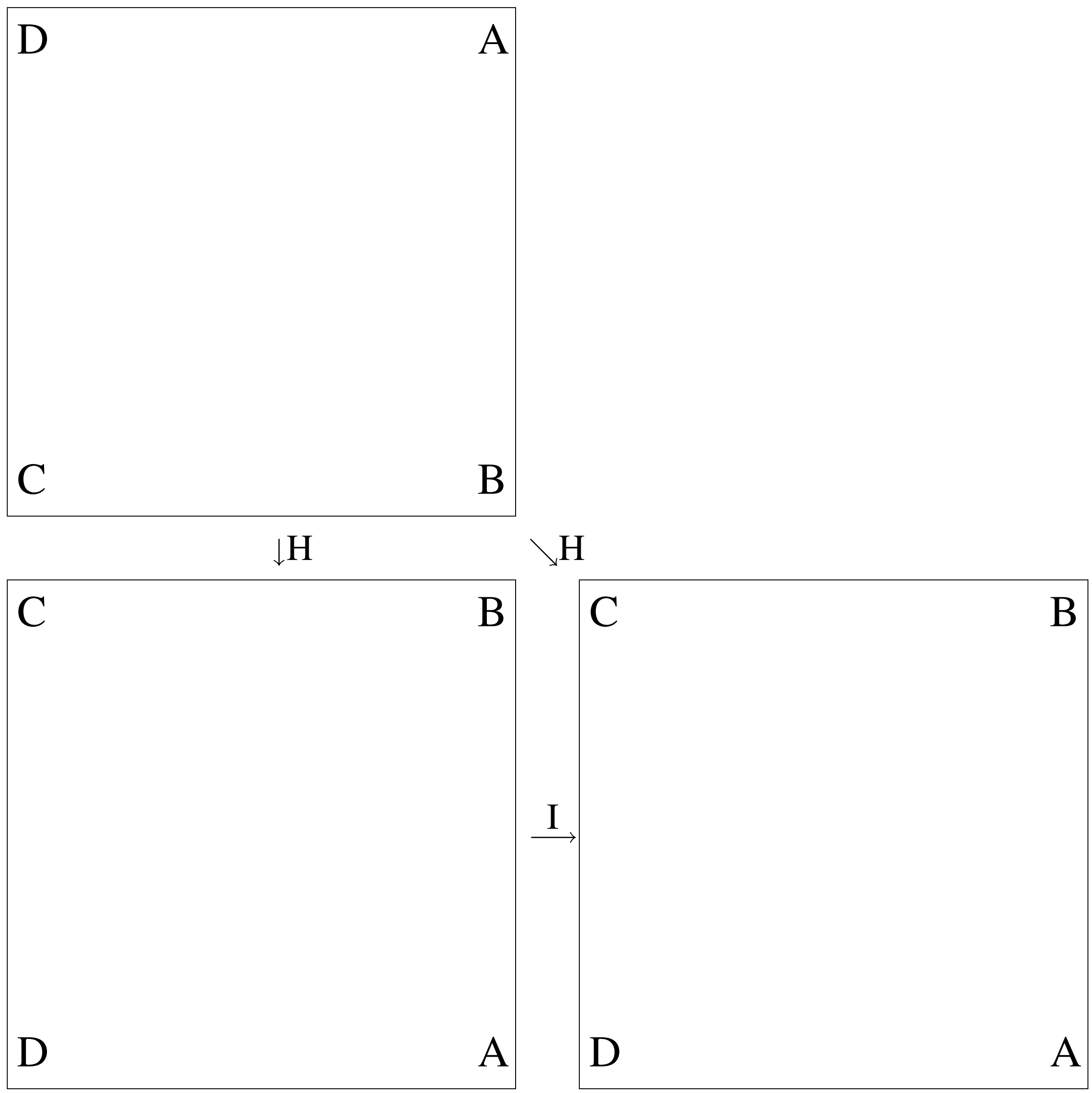
Composing two operations means performing one after another. We write R ∘ H for the result of applying R followed by H.



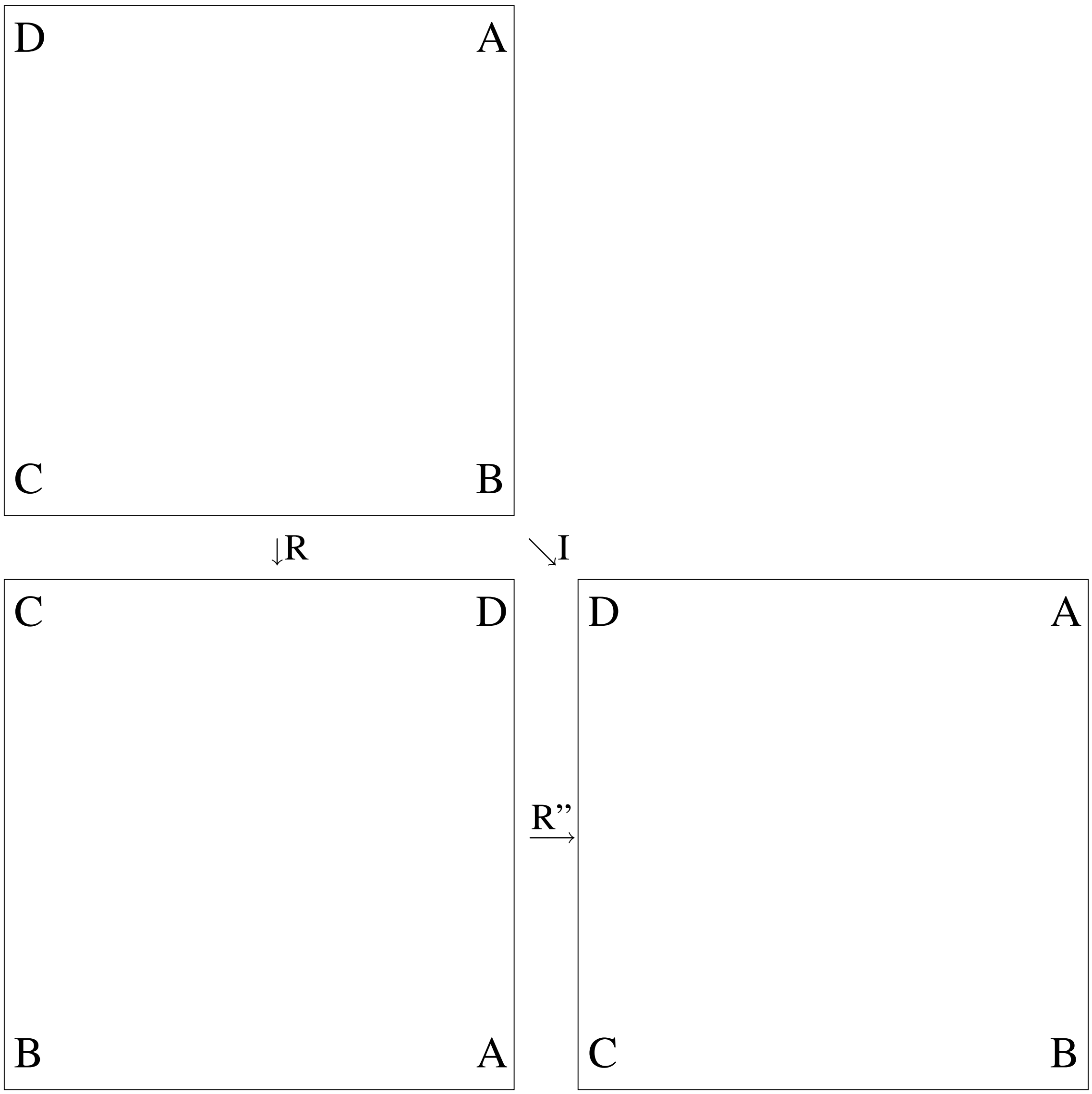
$$R \circ H = D$$



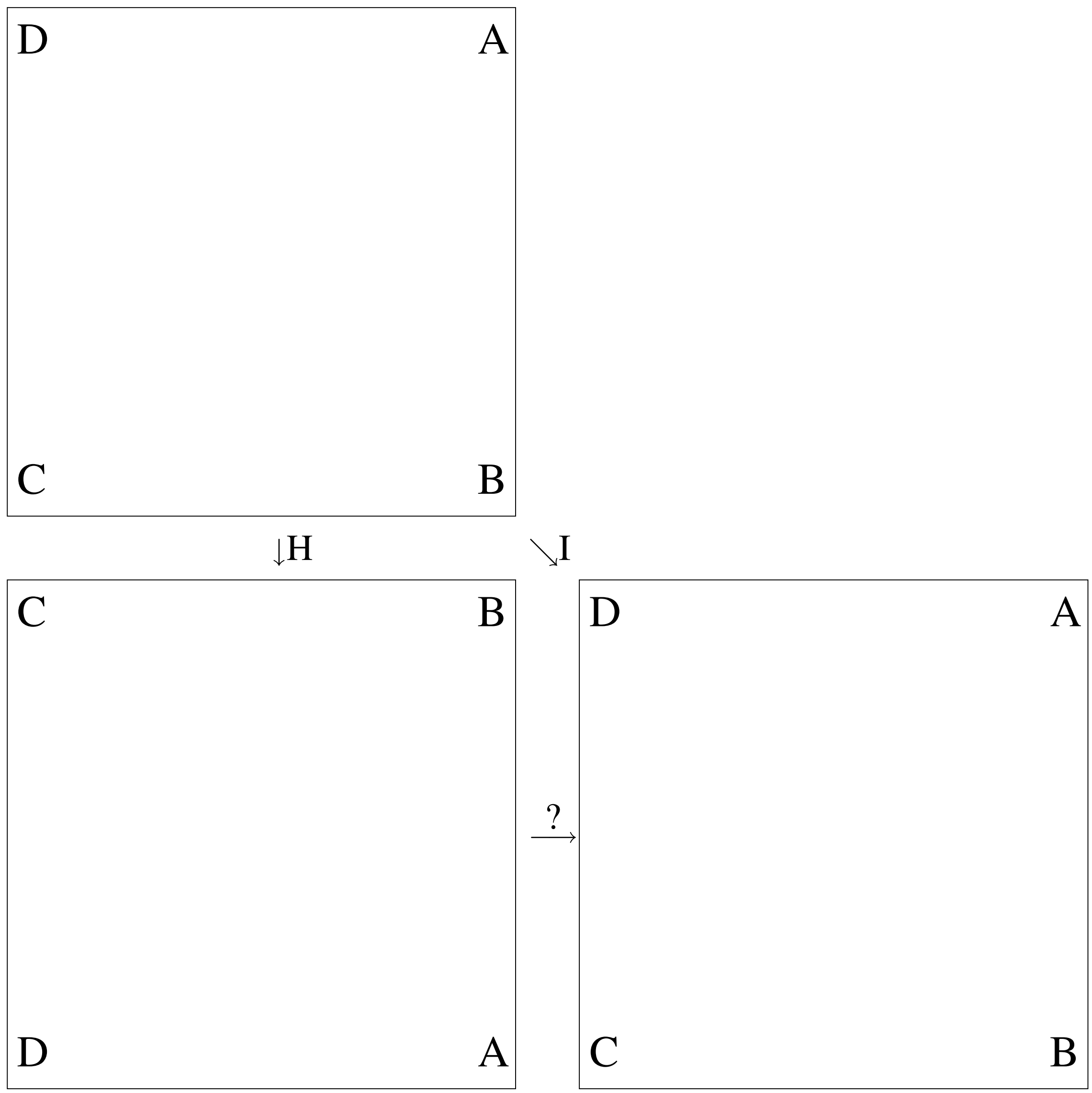
$$H \circ R = D'$$
$$H \circ R \neq R \circ H$$



Identity Element



$R \circ R" = I$
Inverse Element

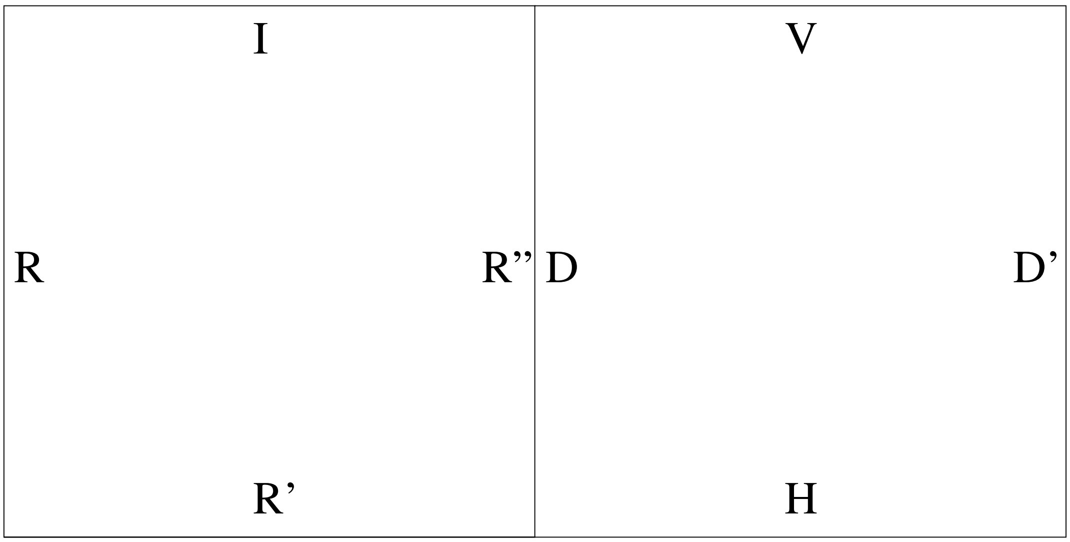


$H \circ ? = I$

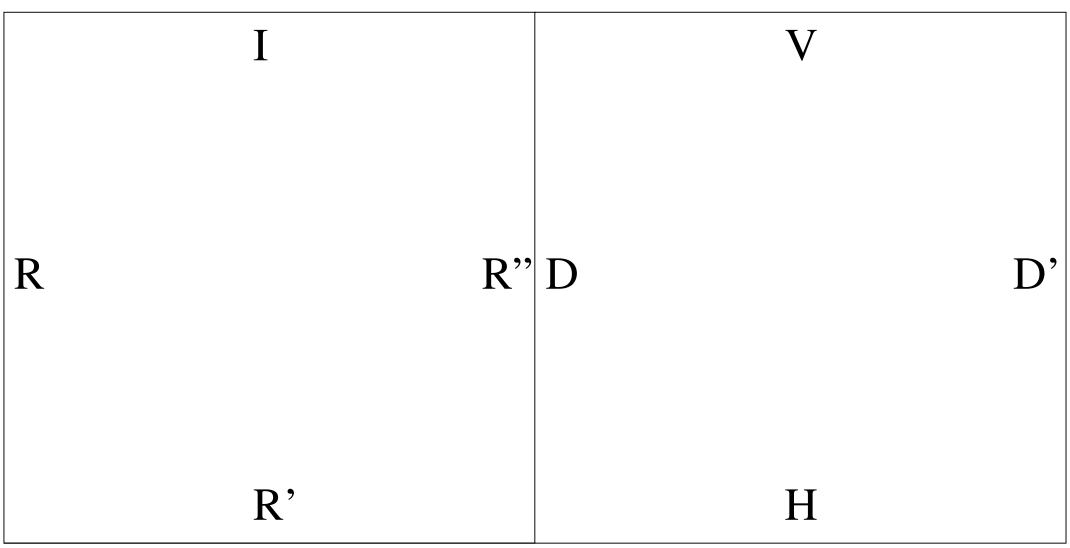
What is the inverse of H?

2 Your own square

Cut out and fold on solid vertical line! After folding D gets superimposed on R”, D’ on R:



Another copy:



Exercises:

1. Compute the following using the square provided.

$R \circ R'$
$R' \circ R'$
$H \circ R'$
$V \circ D$
$H \circ V$

3 The Group

The elements of the group are the rotations and reflections. Call this set **G**:

1. R: Rotate 90.
2. R': Rotate 180.
3. R": Rotate 270.
4. I: Rotate 360
5. V: Reflect around vertical axis.
6. H: Reflect around horizontal axis.
7. D: Reflect around $x = y$ diagonal
8. D': Reflect around $x = -y$ diagonal

The operation is composition ($\circ$).

Theorem 3.1. *Grouphood of Symmetries of Square*
The set **G** *of rotations and reflections forms a group under the operation of composition.*

1. *Closure. The composition operation on G is closed. Verify by inspection. Cut out your squares and check. This means filling out the following operation table (also called a Cayley table):*

	R	*R'*	*R"*	*I*	*V*	*H*	*D*	*D'*
R								
R'								
R"								
I								
V								
H								
D								
D'								

2. *Identity. I is the identity element. That it satisfies the identitu axiom for groups can be seen by inspecting the following portion of the above table:*

	R	*R'*	*R"*	*I*	*V*	*H*	*D*	*D'*
R				*R*				
R'				*R'*				
R"				*R"*				
I	*R*	*R'*	*R"*	*I*	*V*	*H*	*D*	*D'*
V				*V*				
H				*H*				
D				*D*				
D'				*D'*				

3. *Inverse. Each element of* **G** *has an inverse. Verify and show the inverses.*

4. *The operation of composition is associative. Verifying this completely is* very *tedious. Pick a set of 3 elements at least one of which is a reflection (V, H, D or D') and verify associativity for all possible combinations using those 3 elements. Note: There are six equalities to check, corresponding to 6 possible orderings of 3 things. For example, if the elements are a, b, and c, one of the six equalities to check is that*

$$(b \circ a) \circ c = b \circ (a \circ c)$$

5. *We write x^2 as a shorthand for composing an element with itself, that is $x \circ x$. For example:*

$$I^2 = I \circ I = I$$

How many "square roots" does I have and what are they? Obviously there is at least one, one, I itself.