Review

Linear Mixed Model

$$\underset{\sim}{Y} = X\underset{\sim}{\beta} + Z\underset{\sim}{u} + \underset{\sim}{\varepsilon} \qquad (2.2)\ (2.5)$$

where. $X_{n\times p}$ is fixed effects design matrix, $\underset{\sim}{\beta}_{p\times 1}$ fixed effects
$Z_{n\times q}$ is random effects design matrix, $\underset{\sim}{u}_{q\times 1}$ random effects.

$\underset{\sim}{u}_{q\times 1}$ $N(0, D)$
$\underset{\sim}{\varepsilon}_{n\times 1}$ $N(0, R)$ w/ u_i & ε_i independent

$$E(\underset{\sim}{Y}) = X\underset{\sim}{\beta}$$
$$Var(\underset{\sim}{Y}) = Z D Z' + R = V$$

Gaussian LMM

$$\underset{\sim}{Y} \sim N(X\underset{\sim}{\beta}, V) \text{ where } V(\underset{\sim}{\theta}) = ZDZ' + R \qquad (2.8)\ (2.9)$$

joint pdf of $\underset{\sim}{Y}$

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \frac{1}{(2\pi)^{n/2}|V|^{1/2}} \exp\left\{-\frac{1}{2}(\underset{\sim}{Y}-X\underset{\sim}{\beta})' V^{-1}(\underset{\sim}{Y}-X\underset{\sim}{\beta})\right\} \qquad (2.10)$$

$$\log L(\beta,\theta) = \text{constant} - \frac{1}{2}\log(|V|) - \frac{1}{2}(\underset{\sim}{Y}-X\underset{\sim}{\beta})' V^{-1}(\underset{\sim}{Y}-X\underset{\sim}{\beta})$$

(1) $$\frac{\partial \log L}{\partial \beta} = X'V^{-1}\underset{\sim}{Y} - X'V^{-1}X\underset{\sim}{\beta}$$

(2) $$\frac{\partial \log L}{\partial \theta_L} = \frac{1}{2}\left\{(\underset{\sim}{Y}-X\underset{\sim}{\beta})' V^{-1}\frac{\partial V}{\partial \theta_L} V^{-1}(\underset{\sim}{Y}-X\underset{\sim}{\beta}) - \text{trace}\left(V^{-1}\frac{\partial V}{\partial \theta_i}\right)\right\}$$

$L = 1, \ldots, q$

REML

1. Find $K_{n \times (n-p)}$ st $\text{rank}(K) = n-p$ & $K'X = 0$

2. Let $W = K'Y$ & $W \sim N(0, K'VK)$

$$f_W(w) = \frac{1}{(2\pi)^{(n-p)/2} |K'VK|^{1/2}} \exp\left\{ -\frac{1}{2} W'(K'VK)^{-1} W \right\}$$

3. Variance components estimation is ML estimation to normal density

$$(3) \quad \frac{\partial \log L}{\partial \theta_c} = \frac{1}{2}\left\{ Y' P \frac{\partial V}{\partial \theta_c} P Y - \text{trace}\left(P \frac{\partial V}{\partial \theta_c} \right) \right\}, \quad c = 1, \ldots, q$$

where $P = K(K'VK)^{-1}K'$

Note: (3) is a function of $\underset{\sim}{\theta}$ only, but once $\hat{\underset{\sim}{\theta}}$ is obtained can still estimate $\underset{\sim}{\beta}$

ML & REML Estimators

$\hat{\underset{\sim}{\beta}} = (X'V^{-1}X)^{-1} X'V^{-1} Y$ (can use generalized least squares)

$\hat{\underset{\sim}{\theta}}$ by setting (2) & (3) $= 0$ and solving for $\underset{\sim}{\theta}$

p26 * REML is often preferred to ML estimation, because it produces unbiased estimates of covariance parameters by taking into account the loss of degrees of freedom that results from estimating the fixed effects in $\underset{\sim}{\beta}$