In how many different ways can you place X’s in six cells of this 3x3 grid so that there is at least one X in each row?

Solution:

There are two ways to solve this problem; the first in a more intuitive construction and the second relying on combinatorial based counting.

We are placing six X’s in a box containing nine available spaces, with the restriction that each row must contain at least one X. There are two ways of doing this: first, each row contains two X’s and second, one row contains one X, another row contains two X’s and the last row contains three X’s. We simply count the total possible arrangements in each type and add them together.

Type one, first note that for a given row, we can place two X’s in three ways. Either in the first and second column, the second and third column or in the first and third column. Since there are three rows that we do this for, there are $3^3 = 27$ possible arrangements when each row contains two X’s.

Type two, now we are dealing with one row containing one X, another with two X’s and the last with three X’s. We can place the one X in one of the three possible rows. Then, we can place the two X’s in one of the two remaining rows. Finally, we can place the three X’s in one row (the only remaining row). Hence, there are $3! = 3 \times 2 \times 1 = 6$ arrangements. But, the row with one X, we can place the X in column one, two or three, so there are three possible ways to place one X. Similarly to the type one rows, we can place two X’s three different ways as well. Therefore, there are $3! \times 3^2 = 3 \times 2 \times 1 \times 3 \times 3 = 54$ possible ways for the type two arrangements. Adding the type one and type two, we find $27 + 54 = 81$ possible arrangements.

***Using combinatorial reasoning, we count the total number of arrangements of X’s in the nine boxes and then subtract the number of arrangements that do not satisfy our condition (see combinatorics 'choose' on the internet). The only arrangements that do not satisfy our condition occur when there are two rows each with three X’s in them. This occurs three times, when there are three X’s in the first and second row, second and third or first and third. This results in:

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\binom{9}{6} - 3 = \frac{9!}{6! \times 3!} - 3 = 84 - 3 = 81
\] (1)