

Mathematics

The Study of Patterns and Structures

Introduction

“Mathematics” comes from the Greek work “mathematike,” which means scientific craft. Mathematics is defined as “the systematic treatment of magnitude, relationships between figures and forms, and relations between quantities expressed symbolically” (Stein 884). Many times mathematicians examine several cases looking for patterns or relationships and make general conjectures based on those cases. Once they have made a conjecture, mathematicians prove their conjectures by a detailed argument, showing that each step of the proof is justifiable. Mathematical theorems can be proven beyond all doubt using logical argument.

Goals

In non-technical terms, the goal of mathematicians is to find patterns in quantities and their interrelationships, patterns in shapes, patterns in uncertainty, and patterns in change. Hence, one short description of mathematics is sometimes "the study of patterns." Patterns are defined as a consistent arrangement of numbers, forms, or colors that constitute a recognizable design or relationship with each other such that the observer can ascertain the predictability of their occurrence. Mathematicians also develop symbolic languages to describe new kinds of structures and the ways in which structures interact. Patterns and structures in nature are fundamental to the sciences, so mathematics, as the study of patterns and structures, has also been given the label "queen and servant of the sciences." However, mathematics can go well beyond everyday life and the natural world, with the study of quantities giving birth to abstract structures and relationships, and with the study of shapes leading to ideas in four, five, or more dimensions.

Mathematicians who specialize in topics with obvious practical use are sometimes called "applied mathematicians," whereas those whose main interests do not obviously relate to applications can be called "pure mathematicians." There are, of course, many mathematicians with interests in both the practical and the abstract aspects, so the labels describe the ends of a continuum rather than a dichotomy.

In each of the following disciplines, mathematicians have had to develop unique methods and approaches in order to recognize the underlying patterns and structures.

- Arithmetic—the study of number systems and what can be done to numbers
- Geometry—the study of shapes; often taught largely as a system of proofs
- Algebra—the study and manipulation of unknown quantities in equations, and the study of how to solve those equations and find the unknown quantities
- Trigonometry (largely a high school rather than a university subject)—the study in which shapes (graphs) are turned into equations and equations into shapes (graphs), so that geometry and algebra can be brought together and used to advance each other

- Calculus—the study of the mathematics of the equations and graphs of unknown algebraic quantities that *change* from one moment to the next (like sounds waves and accelerating objects) so that the equations, when graphed, *do not* make the clean simple lines and circles encountered in algebra, geometry and trigonometry
- Probability—the study of the relative possibility that an event will occur
- Statistics—the study of how to generalize from the numerical evidence about a small number of things or people (the “sample”) in order to say something about a larger group (the “population” or “parameter”)

Products

Mathematicians do their work in academic institutions, private research institutes, and even financial, technological, and other businesses. Academic mathematicians write mathematics proofs, propose new problems, and develop new methods. They write articles, which are published in professional mathematics journals and books. Mathematicians who work in financial and business settings use mathematics to enhance the productivity of their companies. Movies such as *A Beautiful Mind* and books such as *Journey through Genius* by Dunham help the public understand and appreciate that “beauty, truth, reality, and abstraction are not limited to painters and poets. These concepts occupy the minds of mathematicians, who often view their subject as an art as well as a science” (Selzer 11: 445).

Process

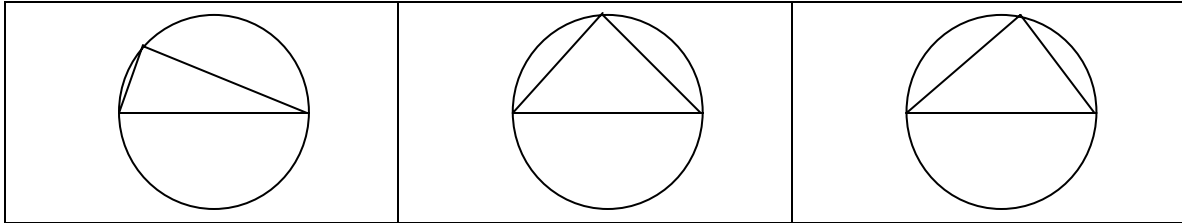
How mathematicians proceed varies from mathematician to mathematician:

- Many times mathematicians work almost exactly as a scientist may. They examine several cases (perhaps with the aid of a computer), make a general conjecture based on those cases, and then gather still more evidence in an effort to test the conjecture. Unlike scientists, mathematicians do not stop at this stage; they proceed to establish their conjecture beyond all doubt by creating a logical, step-wise proof based on previously proven postulates and theorems.
- Another mathematician may proceed quite differently. She may carry out thought experiments about elements in her abstract systems and deductively reason about them, creating new theorems, proposing new topics of investigation, and even creating new methods for working with the elements.
- Sometimes a mathematician will focus on making the mathematics more pleasing intellectually, as in establishing some known result with a simpler or more revealing argument or in relating it to some other area of mathematics in a clever way.

In whatever way they proceed, all mathematicians must first decide what is worthwhile to study. Some mathematicians get ideas from colleagues, others write review chapters or articles in order to become familiar with the literature, and others examine several cases looking for patterns. For example, consider the three triangles below, which are inscribed

in the top half of the circle. Can you see a pattern? Hint: What kind of angle do the two sides of the triangle seem to form at the top of the circle?

Note: This example is an expanded version of the example discussed by William Dunham in *Journey Through Genius* (page 8). See the *Liberal Studies 300: Course Reader*.

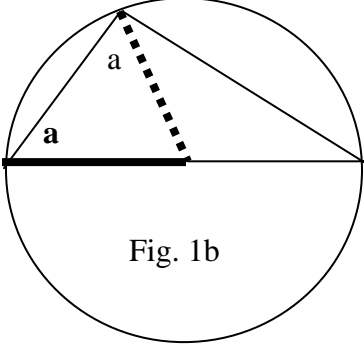
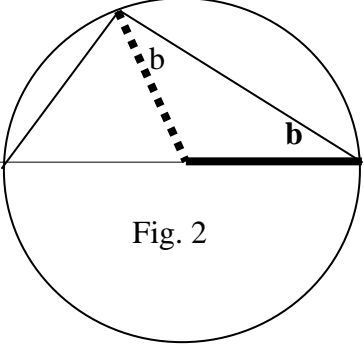
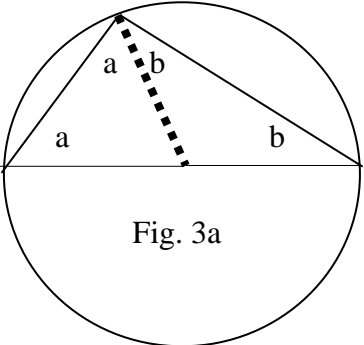
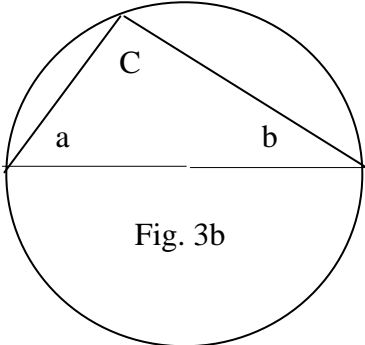


A mathematician named Thales, who lived from about 640 to 546 BC, probably made drawings something like those above, noticed that the angle formed near the top of each circle was a right angle, and proposed the conjecture: “An angle inscribed in a semicircle is a right angle” (Dunham 7). (A conjecture is a tentative proposition formed without sufficient evidence for proof.)

As already stated, a conjecture is not enough. Since the time of Thales, and possibly earlier, mathematicians have used deductive reasoning to write mathematical proofs. (A proof is a step-wise argument built on careful, logical reasoning about the necessary consequences of already known, already proven facts.) In the above example, Thales might have used something like the proof shown below.

If you find the details of the proof hard to follow, just notice how it is set up: There are five steps. In each step something is done which depends upon definitions or previously proven theorems.

<p>Fig. 1a</p>	<p><i>The steps build logically from the initial situation to the final argument that angle C at the top of the original triangle is a right angle.</i></p> <p><i>(See Figure 1a for the original triangle.)</i></p> <p><i>Some Liberal Studies students have done these types of proofs when they studied geometry in high school.</i></p>
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 <p style="text-align: center;">Fig. 1b</p>	<p>Step 1. Draw the line (shown dotted and bold in Figure 1) from the top of the original triangle to the center of the circle. Notice the line (solid and bold), which forms half of the base of the triangle. Both of these lines are radii of the circle. Thus, they have the same length.</p> <p>Step 2. Since the dotted, bold line and the solid, bold line are the same length, the little triangle on the left side of the figure is an “isosceles” triangle. Angles “a” and “a” are thus equal because Thales had previously proven that isosceles triangles have equal angles adjacent to their equal sides.</p>
 <p style="text-align: center;">Fig. 2</p>	<p>Step 3. Repeat steps 1 and 2 for the new triangle in Figure 2 to show that angles “b” and “b” are equal.</p> <p>Step 4. The angle C in Figure 3b is the sum of angles “a” and “b.” See the top of the triangle in Figure 3a.</p> <p>Step 5. From previously proven theorems, Thales knew that the sum of all three angles of a triangle equal two right angles. Thus, for the large triangle in Figure 3b,</p> $ \begin{aligned} 2 \text{ right angles} &= C + a + b && \text{(sum of the three angles)} \\ &= [a + b] + a + b && \text{(substituting for C)} \\ &= 2(a + b) && \text{(combining like terms)} \\ 1 \text{ right angle} &= a + b = C && \text{(dividing by two)} \end{aligned} $ <p>Thus, angle C is a right angle.</p>
 <p style="text-align: center;">Fig. 3a</p>	 <p style="text-align: center;">Fig. 3b</p>

The statement “An angle inscribed in a semicircle is a right angle” has now become a theorem. (A theorem is a proposition that is proven via logical argument based on initial definitions and previously proven propositions and theorems.) The new theorem can be used in future mathematical proofs.

To summarize, mathematicians may or may not use the inductive reasoning of scientists to arrive at a conjecture, but they do not stop there. Mathematicians go on to use the type of logical, deductive reasoning shown above to establish theorems that are true for all time.

Outcomes

Whatever field mathematicians study and however they study it, they want to *understand* it, and quite often the reasoning in a mathematical proof shows why the results hold and on what they depend. Hence, a mathematical proof can both verify a result and also give insight into what makes the result work. Since pure mathematicians have quite often found that results born in abstractions have eventually found applicability in surprising places, they are quite happy to study topics that seem to have relevance only within mathematics itself. Applied mathematicians, in contrast, seek to use the type of insight mentioned above to find ways in which mathematics can be applied in a whole range of different areas.

As stated by Brown and Porter, “It is not generally recognized how much of a part mathematics plays in our daily lives. Some of the mathematics is of course quite old: every day we use numbers, graphs, addition and multiplication. It is easy to forget that the invention of these was at one time a great discovery.” For example, bookkeeping systems were not possible before Arabic numerals replaced Roman numerals. (Think about how difficult it would be to add a column of numbers, such as 11, 8, and 23, if they were written XI, VIII, and XXIII.) Physics, statistics, and engineering depend heavily on modern mathematics, for example error correcting codes in CD players and imaging and category theory in design of the next generation of computer programs and software. Business and economics—from currency trading to budget making to the distribution of goods—are also highly dependent on high-level, often quite abstract mathematical reasoning.

Quality Control

How do mathematicians know that their reasoning is correct? While the following discussion focuses on proofs, the same type of verification, albeit less formally presented, is also used in problem solving and in applied math. In all cases, the key is a logical development of ideas.

How do mathematicians know the reasoning in a proof is correct? Mathematical proofs are written by humans, and humans are fallible. Proofs can be long, even hundreds of pages. If the work is directly related to an application, the proof has a sort of test by seeing whether its prediction holds for the application. Conjecture, like a scientific theory, is strengthened by confirming evidence, but not assured in a mathematical sense (or a scientific sense, for that matter). The result might not hold true for unexamined cases.

How, then, is a mathematical proof of a general result determined to be all right? Mathematical facts most often involve an infinitely large number of cases, so it is not possible to show that the proof is all right by checking even several cases. The usual procedure for evaluating proofs is to have other mathematicians examine the work. A mathematician might give a research talk to colleagues, inviting them to examine her

reasoning. She might circulate the work to other experts in the field, who then examine the work quite critically, seeking errors in reasoning or looking for unstated assumptions that might not always be true. Eventually she would submit the work to a technical journal for consideration for publication. The journal editors then look for experts who review the work anonymously. The author-mathematician is not told who these reviewers are so that the reviewers can be critical without fear of damaging a relationship with someone they might know (most experts in a field know the other experts in that field). The hope is that all of this checking, even though it is done by other humans, catches every shortcoming in the reasoning. Even though it is certainly possible that several experts can miss some error, a mathematical result that survives this review process can usually be counted on to be correct.

While some proofs are very long and complicated, mathematicians ultimately strive for the most concise proof. “The highest compliment that a mathematician can receive about a theorem he has developed is, ‘It is beautiful,’ or ‘It is elegant’” (Selzer 11: 445).

Works Consulted

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