THE FROBENIUS NUMBER OF GEOMETRIC SEQUENCES

Darren C. Ong[1]
Texas Christian University, Fort Worth, Texas, USA
d.ong@tcu.edu

Vadim Ponomarenko[2]
San Diego State, San Diego, California, USA
vadim@sciences.sdsu.edu

Received: , Accepted: , Published:

Abstract

The Frobenius problem is about finding the largest integer that is not contained in the numerical semigroup generated by a given set of positive integers. In this paper, we derive a solution to the Frobenius problem for sets of the form $\{m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k\}$, where m, n are relatively prime positive integers.

1. Introduction

The *Frobenius number* of a set of positive integers $\{a_1, \ldots, a_k\}$ (known as the generators) is the largest integer that is not in the numerical semigroup generated by the generators. This number is denoted by $g(a_1, \ldots, a_k)$. Finding the Frobenius number without any restrictions on the set of generators is known to be NP-hard [1]. However, James Joseph Sylvester discovered a simple formula for the problem with two generators in 1884 [7]. Efficient algorithms for the solution of the three generator case were discovered by Greenberg [3] in 1988. Also of particular interest is a formula by Roberts for the Frobenius number for arithmetic sequences [5], and a formula by Lewin for almost arithmetic sequences [4]. An extensive list of literature on the problem can be found in [2].

In this note, we investigate the Frobenius number for geometric sequences, that is, sequences of the form $\{a, ar, ar^2, \ldots, ar^k\}$ where a is an initial value and r the common ratio. Since $\gcd(a, ar, ar^2, \ldots, ar^k)$ must equal one[6], then we have that $a = m^k$ and $r = n/m$ where m, n are relatively prime integers. Our main result is the following:

[1] Undergraduate Mathematics Major
[2] Assistant Professor

Theorem. *Let m, n, k be integers such that $gcd(m, n) = 1$.*

$$g(m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k) = n^{k-1}(mn - m - n) + \frac{(n-1)m^2(m^{k-1} - n^{k-1})}{(m-n)}.$$

Acknowledgements We are thankful to Stan Wagon for insightful suggestions concerning the exposition of this result and conversations about the Frobenius problem in general. We also wish to thank our anonymous referee.

2.Finding the Frobenius number

In the rest of the paper, we shall denote by $A(m, n, k)$ the numerical semigroup generated by $\{m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k\}$. We will also write $G(m, n, k)$ instead of $g(m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k)$.

Lemma 1. *For m, n relatively prime and $k \geq 1$,*

$$G(m, n, k+1) \geq (n-1)m^{k+1} + nG(m, n, k).$$

Proof. We have to show that $(n-1)m^{k+1} + nG(m, n, k)$ is not in $A(m, n, k+1)$. Assume instead that $(n-1)m^{k+1} + nG(m, n, k) \in A(m, n, k+1)$. Then

$$(n-1)m^{k+1} + nG(m, n, k) = \sum_{i=0}^{k+1} c_i m^i n^{k+1-i}, c_i \in \mathbb{Z}_{\geq 0}$$

Taking both sides $\mod n$ we obtain $-m^{k+1} \equiv c_{k+1}m^{k+1}$. Since m, n are relatively prime, we conclude $c_{k+1} \equiv -1 \mod n$.
Say that $c_{k+1} = bn - 1$ for some positive integer b. Then we have

$$(n-1)m^{k+1} + nG(m, n, k) = \left[\sum_{i=0}^{k-1} c_i m^i n^{k+1-i}\right] + ((b-1)m + c_k)m^k n + (n-1)m^{k+1}$$

and so

$$G(m, n, k) = \left[\sum_{i=0}^{k-1} c_i m^i n^{k-i}\right] + ((b-1)m + c_k)m^k$$

But this implies $G(m, n, k) \in A(m, n, k)$, which is absurd. Thus we conclude that $(n-1)m^{k+1} + nG(m, n, k) \notin A(m, n, k+1)$, and so $G(m, n, k+1) \geq (n-1)m^{k+1} + nG(m, n, k)$. $\square$

Lemma 2. *For m, n relatively prime and $k \geq 1$,*

$$G(m,n,k+1) \leq (n-1)m^{k+1} + nG(m,n,k).$$

Proof. We will show that if $y > (n-1)m^{k+1} + nG(m,n,k)$, then $y \in A(m,n,k+1)$. Let $y \equiv dm^{k+1} \mod n$, $d \in [0, n-1]$. Let $z = y - dm^{k+1}$. Since $z \equiv 0 \mod n$, we have $z = nw$ for some non-negative integer w. But $y > (n-1)m^{k+1} + nG(m,n,k)$ implies $z > nG(m,n,k)$, and so $w > G(m,n,k)$, and thus $w \in A(m,n,k)$. But this means that $y = nw + dm^{k+1} \in A(m,n,k+1)$, and so $G(m,n,k+1) \leq (n-1)m^{k+1} + nG(m,n,k)$ □

Proof of Theorem. By induction on k. For $k = 1$ this reduces to the result of Sylvester in [7], $G(m,n,1) = mn - m - n$. Suppose that it is true for $k = t$ and thus

$$G(m,n,t) = n^{t-1}(mn-m-n) + \frac{(n-1)m^2(m^{t-1} - n^{t-1})}{m-n}.$$

By lemmas 1 and 2 we have

$$\begin{aligned} G(m,n,t+1) &= (n-1)m^{t+1} + n\left(n^{t-1}(mn-m-n) + \frac{(n-1)m^2(m^{t-1}-n^{t-1})}{(m-n)}\right) \\ &= n^t(mn-m-n) + (n-1)\left(m^{t+1} + \frac{nm^2(m^{t-1}-n^{t-1})}{(m-n)}\right) \\ &= n^t(mn-m-n) + \frac{(n-1)m^2(m^t-n^t)}{(m-n)} \end{aligned}$$

which is the theorem for $k = t+1$. Thus the induction holds.

□

References

[1] J.L. Ramírez Alfonsín. Complexity of the Frobenius prolem. *Combinatorica*, 16(1):143–147, 1996.

[2] J.L. Ramírez Alfonsín. *The Diophantine Frobenius Problem.* Oxford University Press, Oxford, 2005.

[3] H.Greenberg. Solution to a linear Diophantine equation for nonnegative integers. *J.Algorithms*, 9:343–353, 1988.

[4] M. Lewin. An algorithm for a solution of a problem of Frobenius. *J. Reine Angew. Math.*, 276:68–82, 1975.

[5] J.B. Roberts. Note on linear forms. *Proc. Amer. Math. Soc.*, 7:465–469, 1956.

[6] E.S. Selmer. On the linear diophantine problem of Frobenius. *J. Reine Angew. Math.*, 293/294:1–17, 1977.

[7] J.J. Sylvester. Question 7382. *Mathematical Questions from the Educational Times*, 41:21, 1884.