Increased Regions of Stability for a Two-Delay Differential Equation

UBC – IAM Seminar

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Fall 2013

Outline

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- **Collaborators**
  - Paul Zak (CGU), NSF REU undergraduate at SDSU
  - Timothy Buskin, Master's thesis at SDSU

Introduction

Two-Delay Differential Equation

$$\dot{y}(t) + A\,y(t) + B\,y(t-1) + C\,y(t-R) = 0$$

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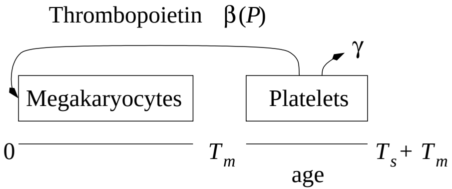
  - E. F. Infante noted an odd stability property observed in a two delay economic model
  - Multiple delays are important for biological models
  - Developed special geometric techniques for analysis of delay equations

Platelet Model 1

Two-delay Model for Platelets (Bélair and Mackey, 1987)



$$\frac{dP}{dt} = -\gamma P(t) + \beta(P(t - T_m)) - \beta(P(t - T_m - T_s))e^{-\gamma T_s}$$

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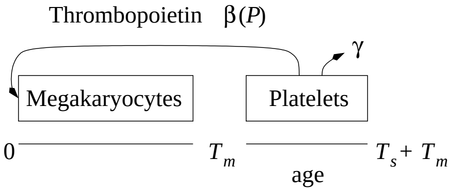

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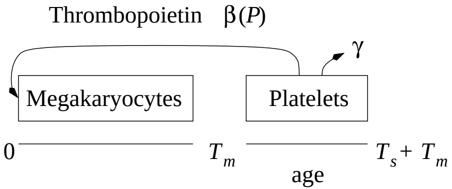

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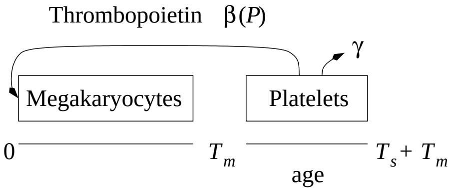

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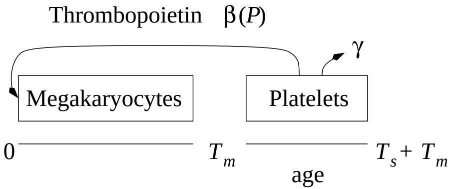

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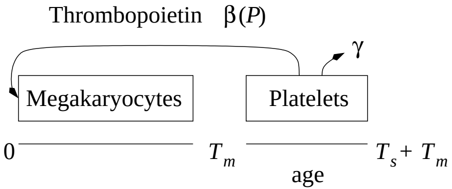

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Modified Platelet Model 2

Modified Platelet Model

- Examine a modified form:

$$\frac{dP}{dt} = -\gamma P(t) + \frac{\beta_0 \theta^n P(t-R)}{\theta^n + P^n(t-R)} - f \cdot \frac{\beta_0 \theta^n P(t-1)}{\theta^n + P^n(t-1)}$$

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- Introduced parameter f, which is different
- Wanted a scaling factor, instead of time delay varying discount

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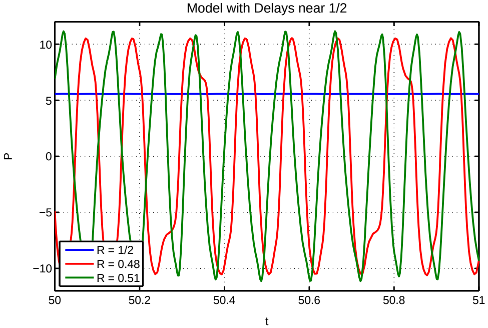

Figure shows stability at $R = \frac{1}{2}$, but irregular oscillations for delays nearby

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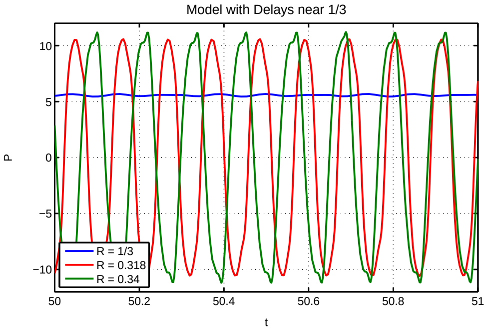

Figure shows stability at $R = \frac{1}{3}$, but irregular oscillations for delays nearby (Same parameters as $R = \frac{1}{2}$)

DDE with One Delay

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$$\dot{y}(t) = ay(t) + by(t - r)$$

This is an ∞-dimensional problem (time history)

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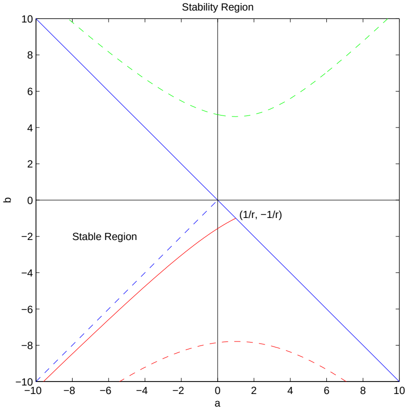
Create distinct curves $\omega \in \left(\frac{(n-1)\pi}{r}, \frac{n\pi}{r}\right)$ for $n = 1, 2, \ldots$

Stability Region - DDE with One Delay



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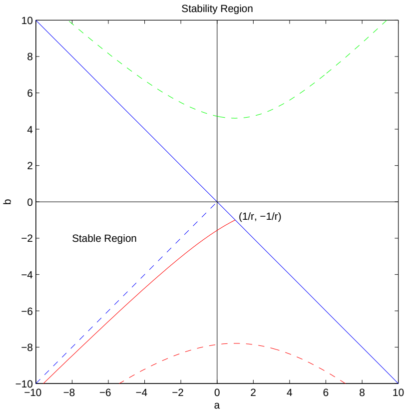

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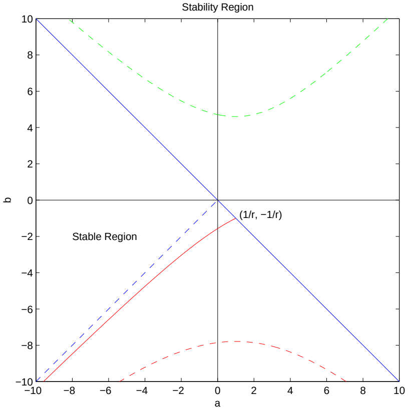

- Real root crossing solid blue line ($a + b = 0$)
- Hopf bifurcation crossing solid red line

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- Imaginary root crossings are distinct, non-intersecting curves

Linear Two-Delay Differential Equation

Scalar Linear Two-Delay Differential Equation

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Theorem (Minimum Region of Stability (MRS))

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- We examine the stability region in BC-plane for fixed A relative to the MRS

Bifurcation Curves and Surfaces

Study the **Characteristic Equation** at $\lambda = i\omega$ or

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Bifurcation Surface j, Λ_j, (***Bifurcation Curve j***, Γ_j) is determined by:

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- The bifurcation curves for the 1-delay DE were non-intersecting

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The **Characteristic Equation** gives a real root crossing at $\lambda = 0$, so $A + B + C = 0$

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- This plane is always part of the boundary of the stability surface
- This plane lies along one edge of the MRS
- Degenerate equilibrium solutions, $y_e(t) = k$, are along this surface

First Bifurcation Curve, Γ_1

Our geometric approach begins with small values of A

The 1^{st} **Bifurcation Curve,** Γ_1 starts as $\lambda = i\omega \to 0$ intersecting Λ_0 along the line

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Theorem (Starting Point - Mahaffy, Joiner, Zak)

If $R > R_0 \approx 0.0117$, then the stability surface comes to a point at $(A_0, B_0, C_0) = \left(-\frac{R+1}{R}, \frac{R}{R-1}, \frac{1}{R(1-R)}\right)$, and the DDE (1) is unstable for $A < A_0$.

Early Bifurcation Surface 1

- The bifurcation surface begins at the starting point (for $R > R_0$)

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- Λ_2 self-intersects for $A \in [A_2^p, A_1^*]$, where A_1^* is the A-value that this **Stable Spur** joins the main stability surface
- These **stable spurs** and **transition** values are key to understanding the asymptotic structure of the stability region

SDSU

Early Bifurcation Surface 2

Definition (Stability Spur)

If Bifurcation Surface $j+1$ self-intersects above the zero-root crossing plane as A increases, with the **Cusp Point of Spur j** denoted A_j^p, then the quasi-cone-shaped **stability spur** has its cross-sectional area monotonically increase with A until A reaches a transitional value, A_j^*. The one-dimensional distance $A_j^* - A_j^p$ is the **Spur j's length**.

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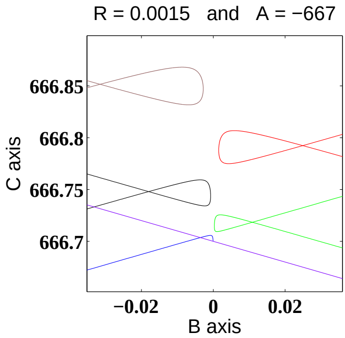
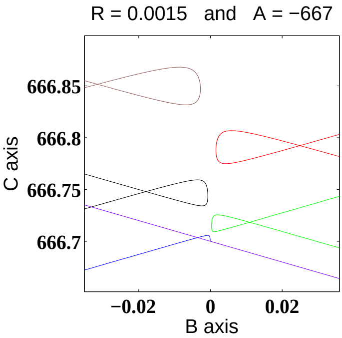
- In BC cross-sectional regions, the **Stability Spurs** produce disconnected regions of stability
- The complete ABC 3D stability surface has been proven to be connected
- Significantly, a **Stability Spur** can draw the stability region away from the main stability surface before attaching

Early Bifurcation Surface



Early Bifurcation Surface 3



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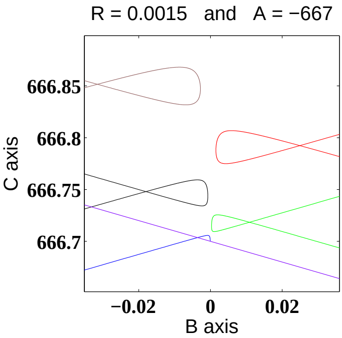

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- $R = 0.0015$ is very small and is below our primary area of study

Early Bifurcation Surface 4

Definition (Transition)

There are critical values of A corresponding to where $B(\omega)$ and $C(\omega)$ become indeterminate at $\omega = \frac{j\pi}{1-R}$. These **transitional** values of A, denoted by A_j^*, satisfy

$$A_j^* = -\left(\frac{j\pi}{1-R}\right)\cot\left(\frac{jR\pi}{1-R}\right), \; j = 1, 2, \ldots .$$

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- **Transitions** are where **Stability Spurs** join the main region of stability
- These **Transitions** significantly **enlarge** the stability region

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Definition (Transition)

There are critical values of A corresponding to where $B(\omega)$ and $C(\omega)$ become indeterminate at $\omega = \frac{j\pi}{1-R}$. These **transitional** values of A, denoted by A_j^*, satisfy

$$A_j^* = -\left(\frac{j\pi}{1-R}\right)\cot\left(\frac{jR\pi}{1-R}\right), \quad j = 1, 2, \ldots .$$

- **Transitions** are where **Stability Spurs** join the main region of stability
- These **Transitions** significantly **enlarge** the stability region
- When R rational, $A_j^* \to +\infty$ for some j

Early Bifurcation Surface

- At a **transition**, Γ_j and Γ_{j+1} coincide at the specific point (B_j^*, C_j^*), where

$$B_j^* = (-1)^j \frac{(1-R)\cos\left(\frac{jR\pi}{1-R}\right) - jR\pi\csc\left(\frac{jR\pi}{1-R}\right)}{(1-R)^2}$$

$$C_j^* = -(-1)^j \frac{(1-R)\cos\left(\frac{j\pi}{1-R}\right) - j\pi\csc\left(\frac{j\pi}{1-R}\right)}{(1-R)^2}$$

Early Bifurcation Surface 5

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- Transitions create a **Degeneracy Line**, defined Δ_j, that parallels one of the boundaries of the MRS

Early Bifurcation Surface

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- All along the **Degeneracy Line**, Δ_j,

$$(B - B_j^*) + (-1)^j (C - C_j^*) = 0, \qquad A_j^*,$$

there are two roots of the characteristic equation on the imaginary axis with $\lambda = \frac{j\pi}{1-R} i$

Early Bifurcation Surface 5

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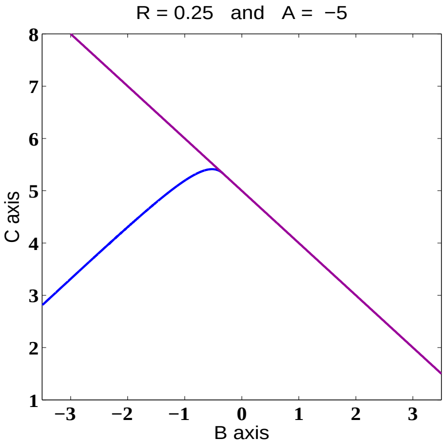
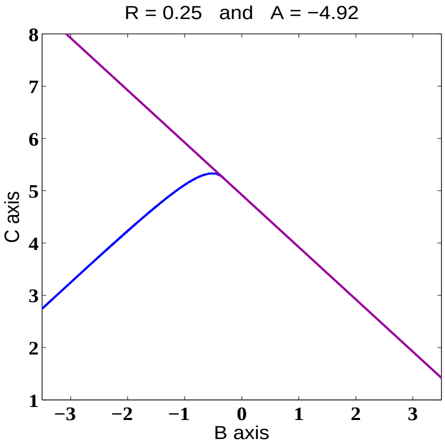
- The next slides show an animation of the early **stability surface** as A increases from A_0 to A_1^* for $R = \frac{1}{4}$

Early Stability Surface



Early Stability Surface



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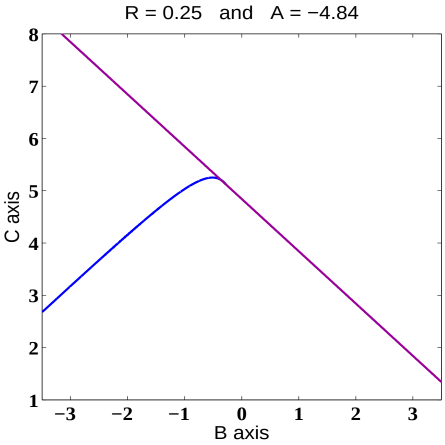

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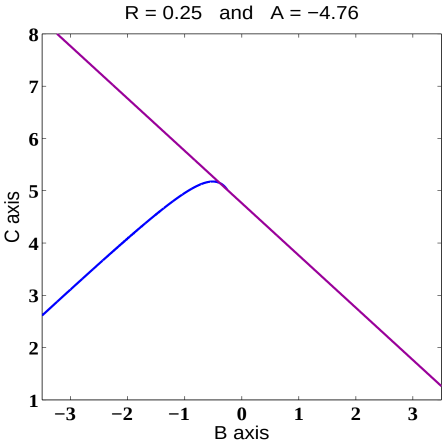

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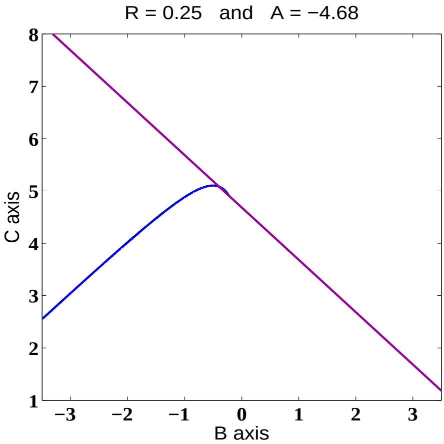

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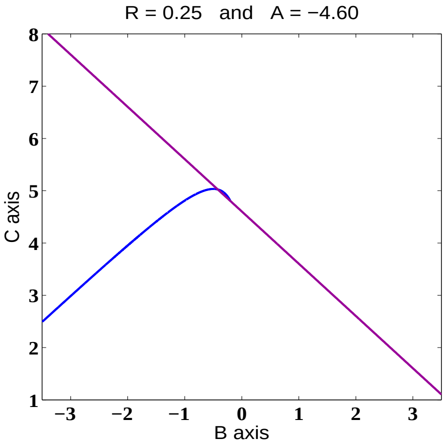

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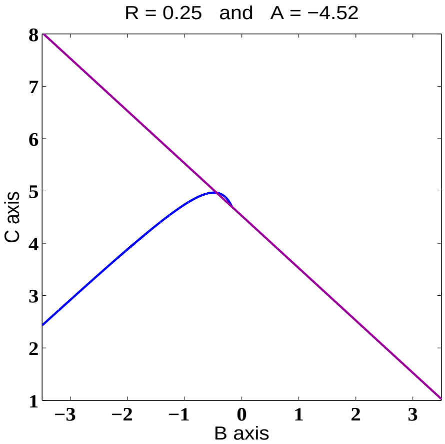

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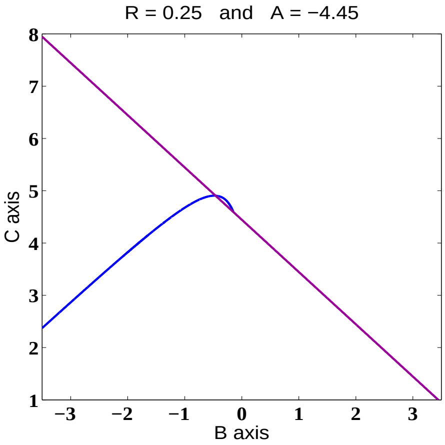

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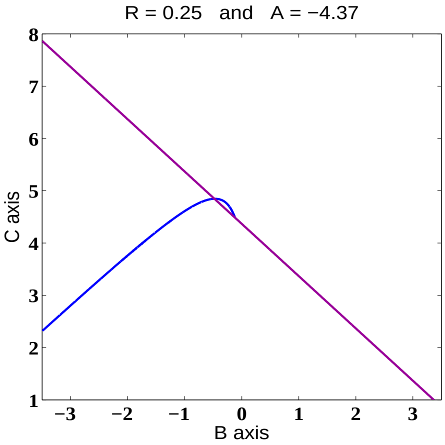

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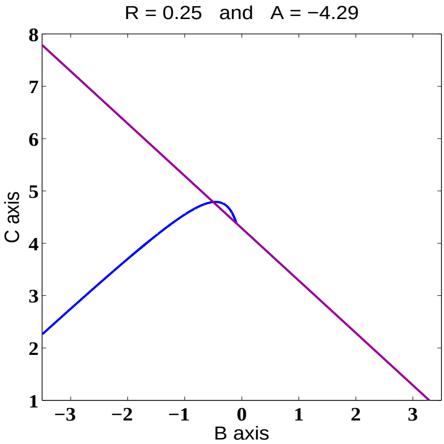

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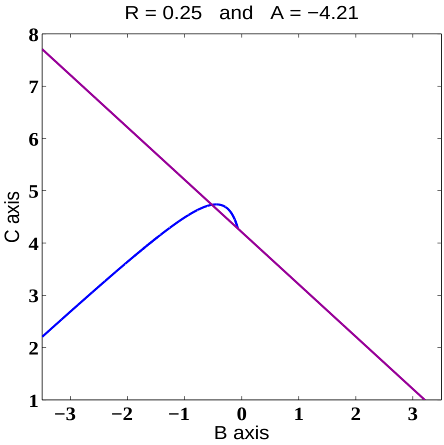

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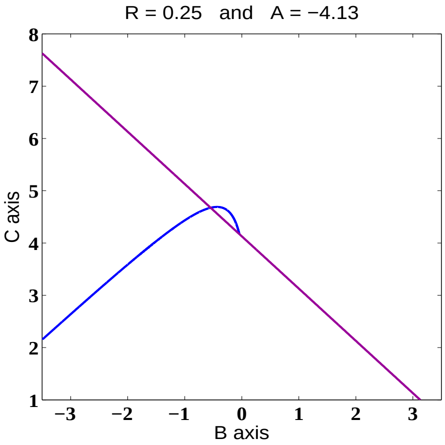

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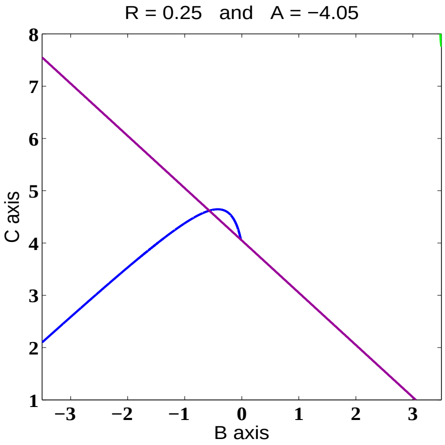

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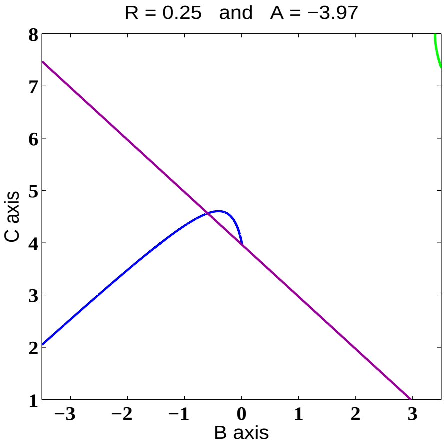

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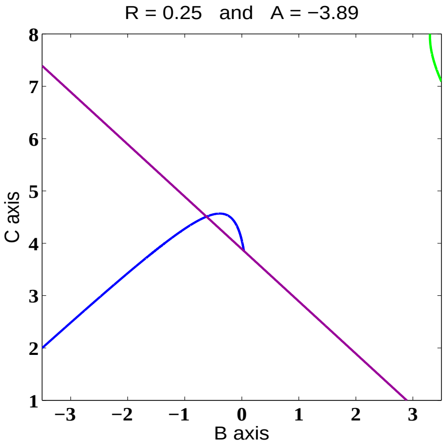

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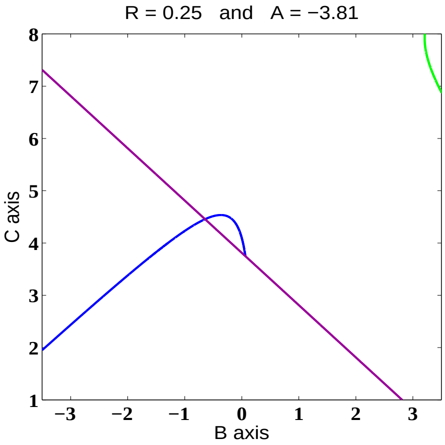

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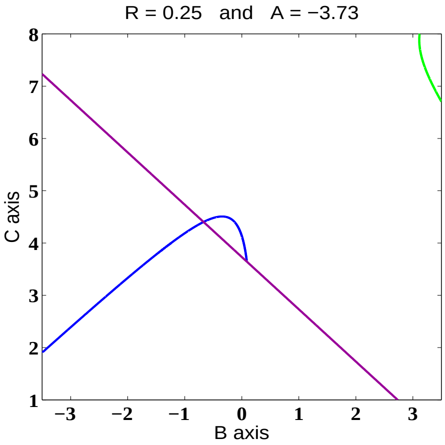

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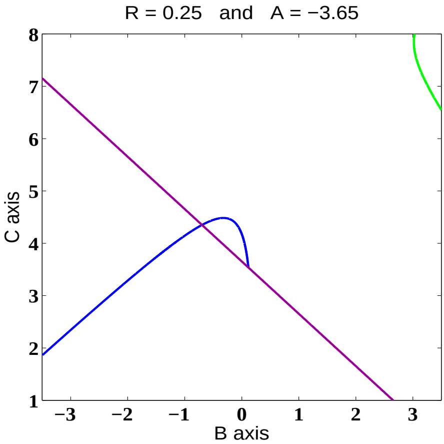

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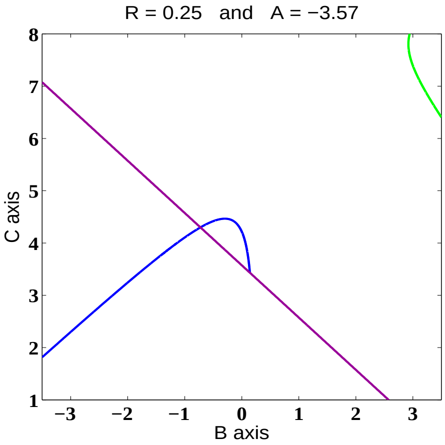

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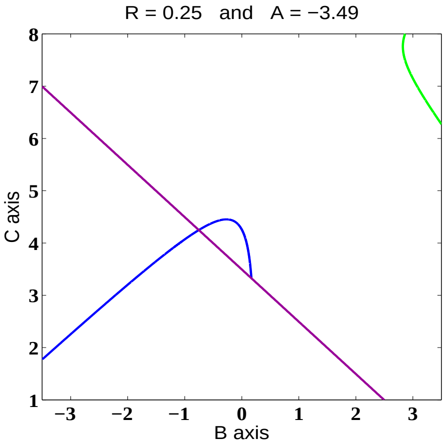

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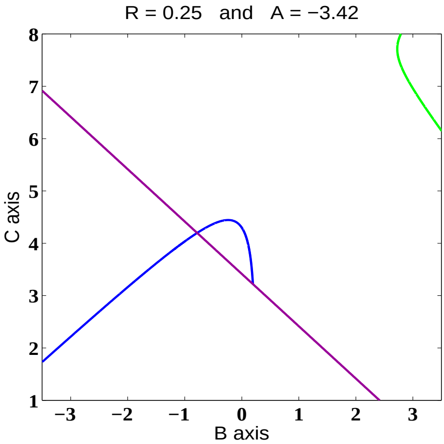

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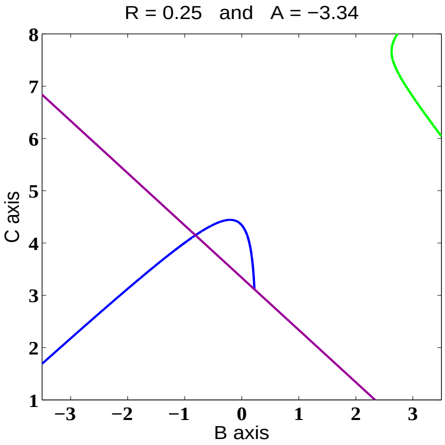

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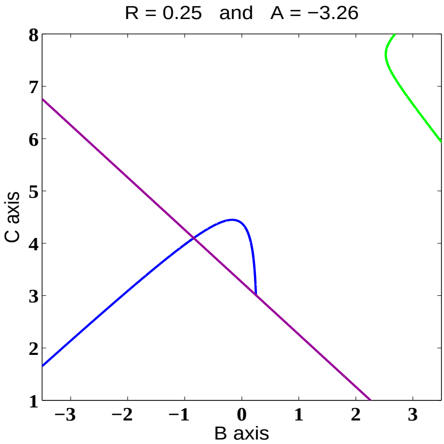

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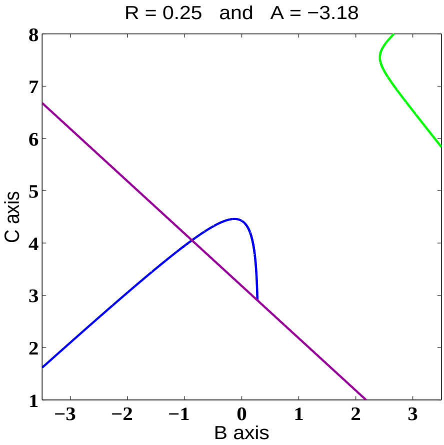

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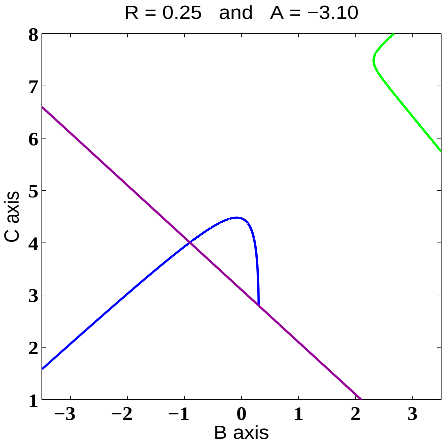

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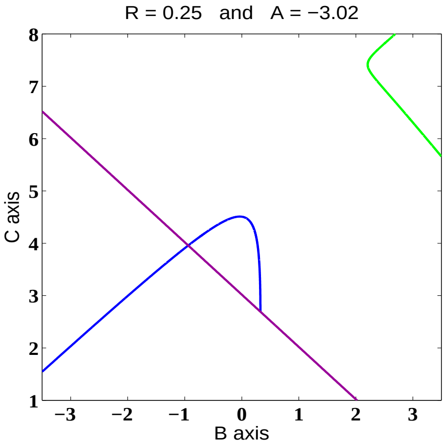

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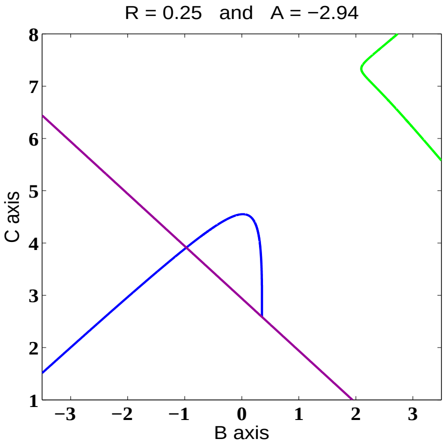

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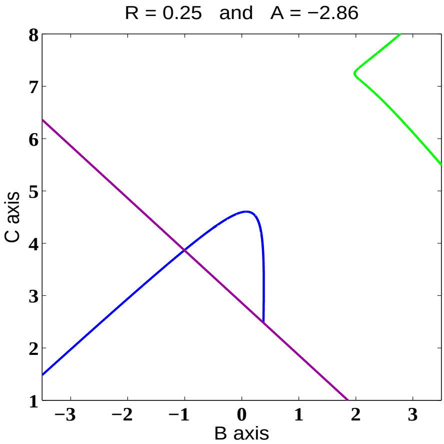

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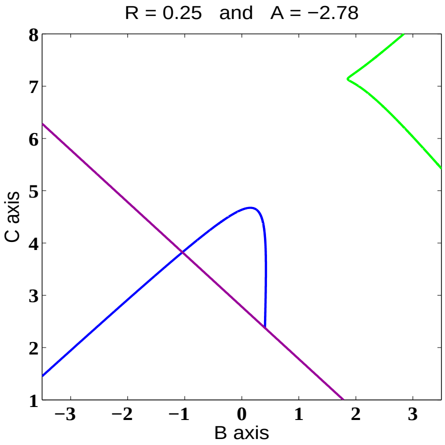

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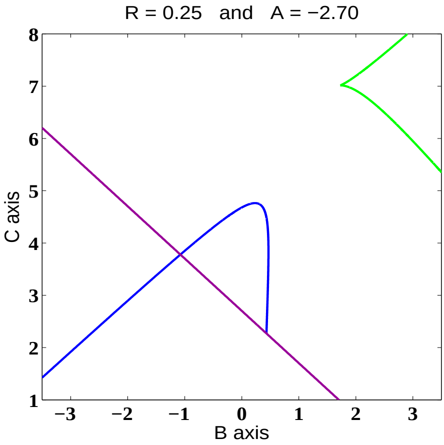

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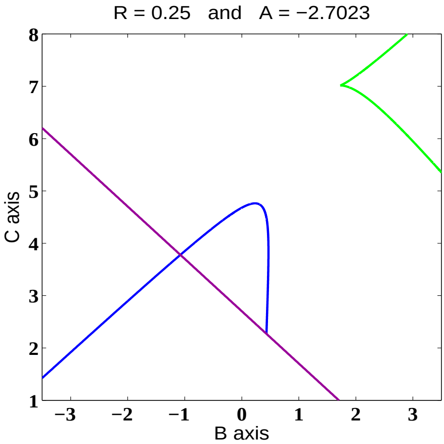

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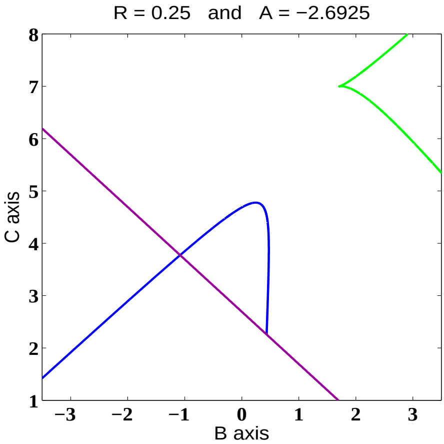

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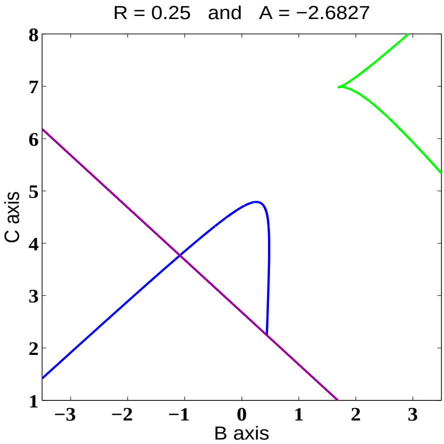

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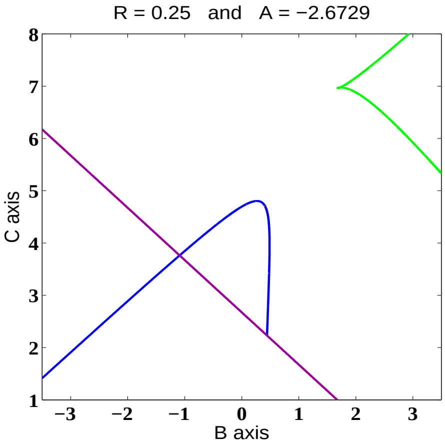

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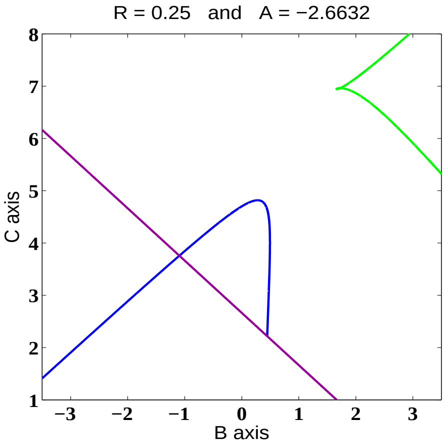

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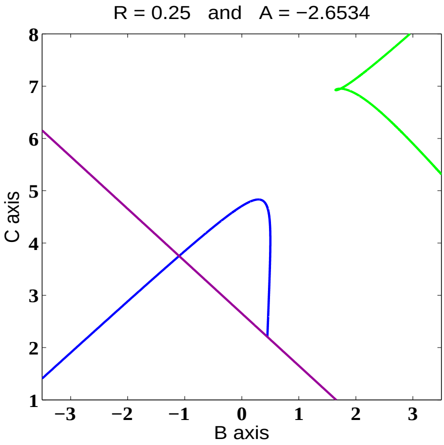

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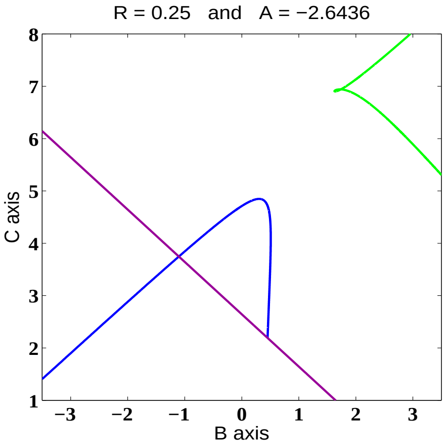

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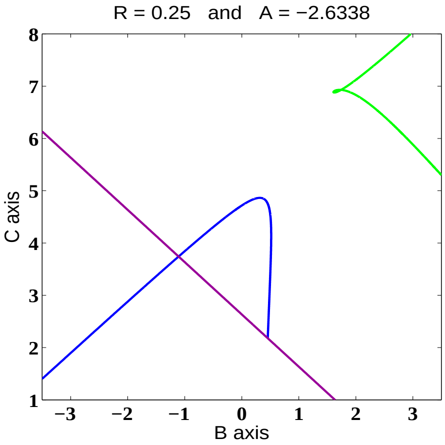

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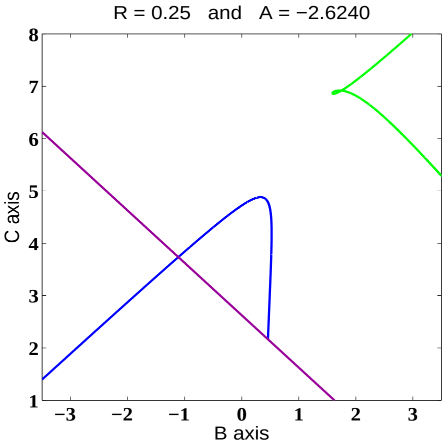

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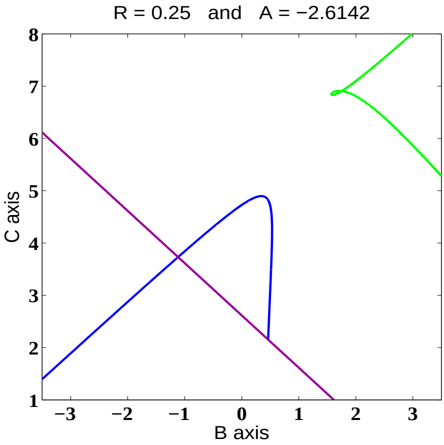

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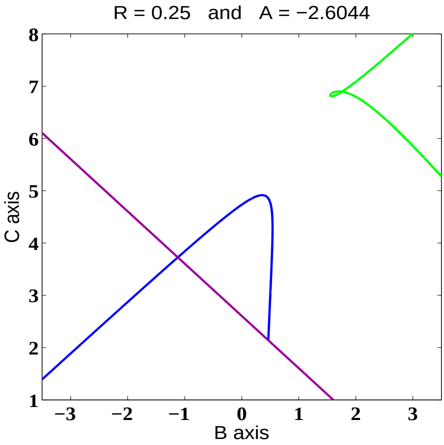

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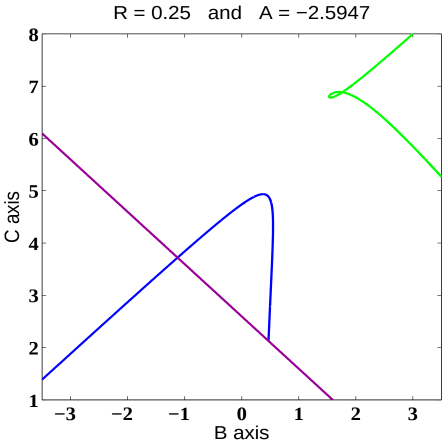

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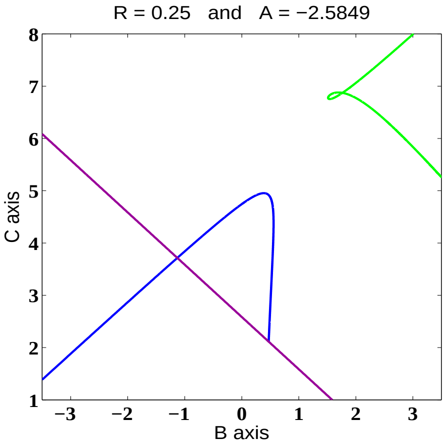

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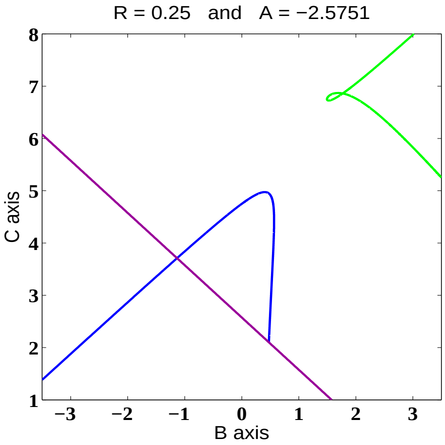

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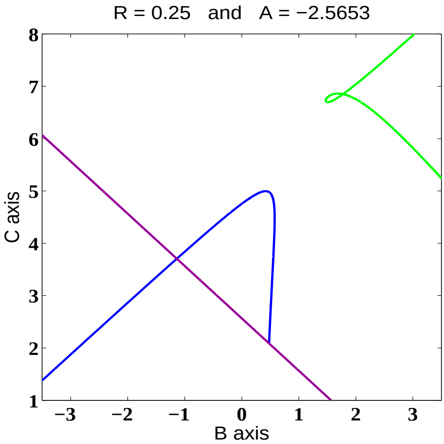

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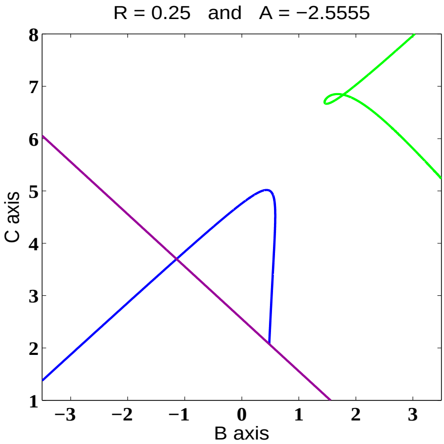

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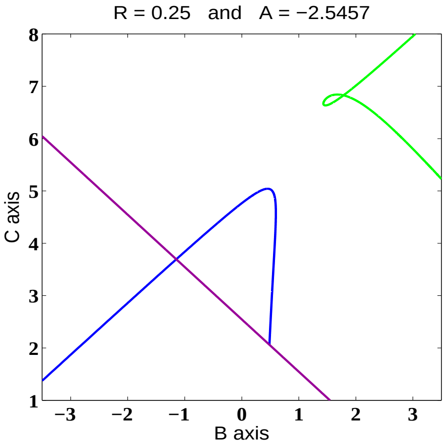

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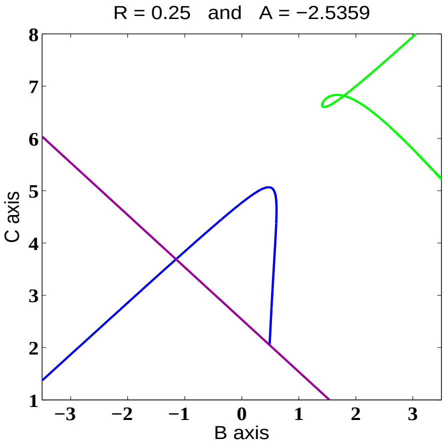

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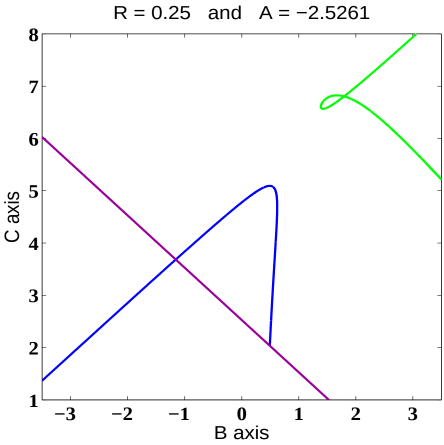

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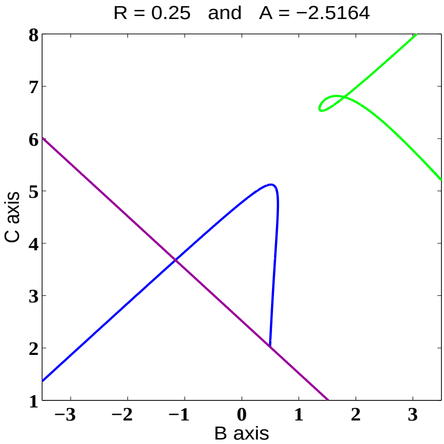

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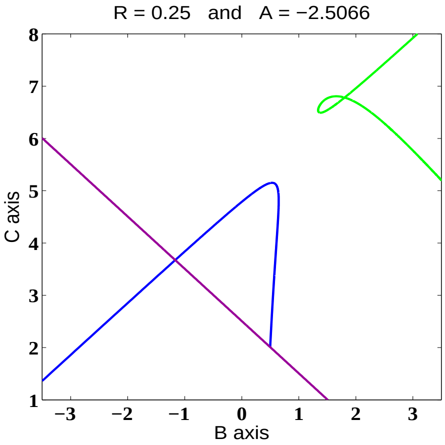

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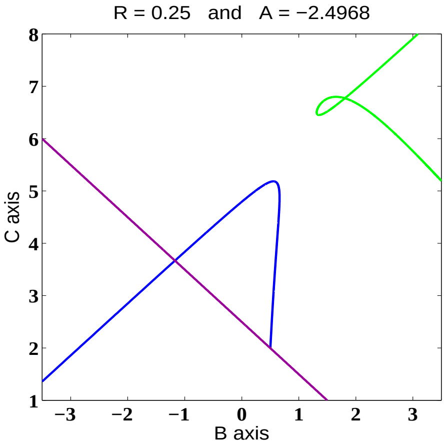

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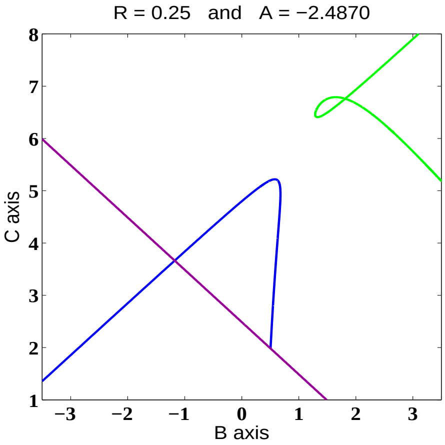

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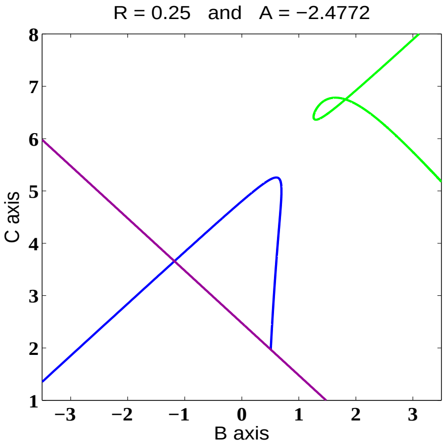

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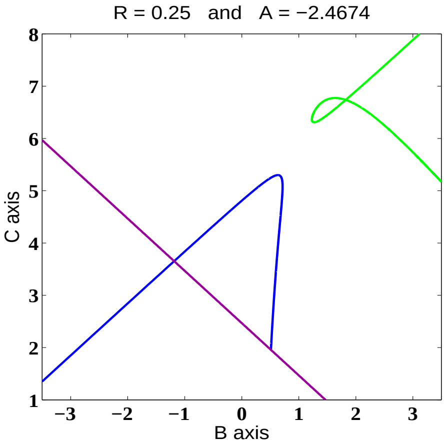

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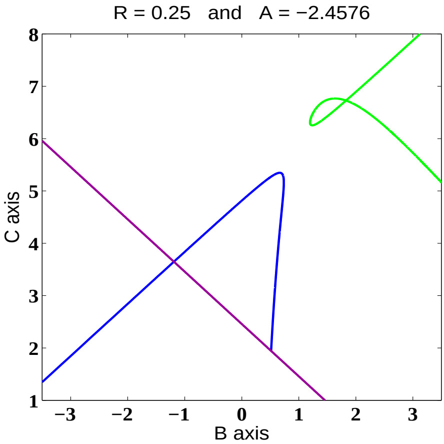

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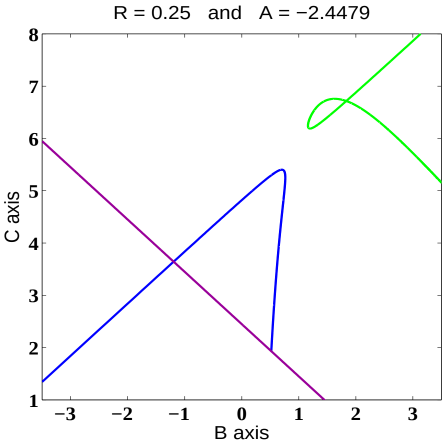

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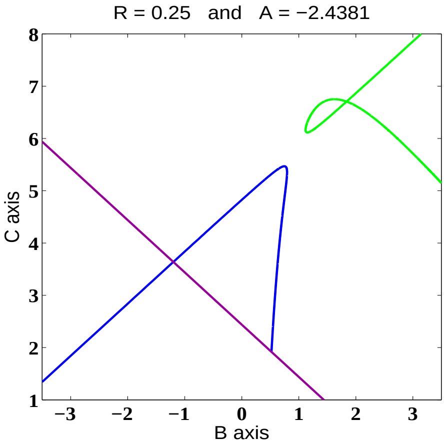

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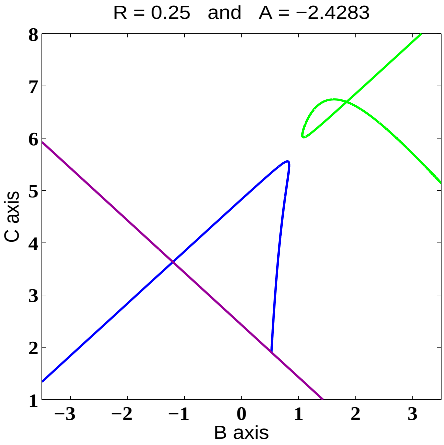

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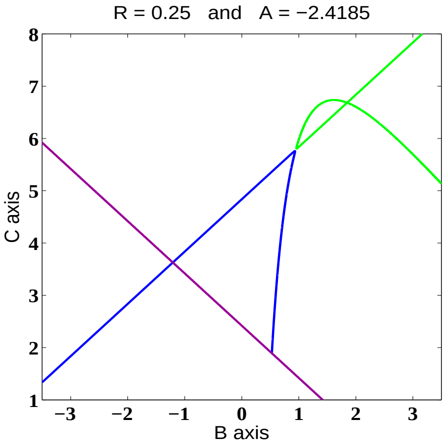

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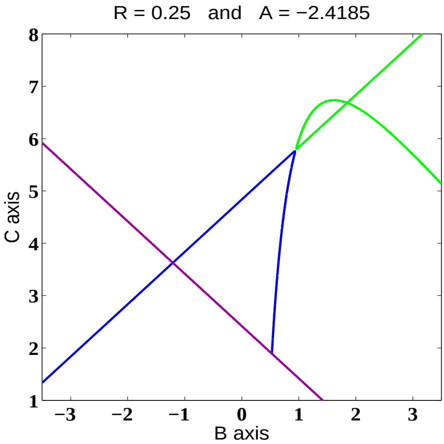

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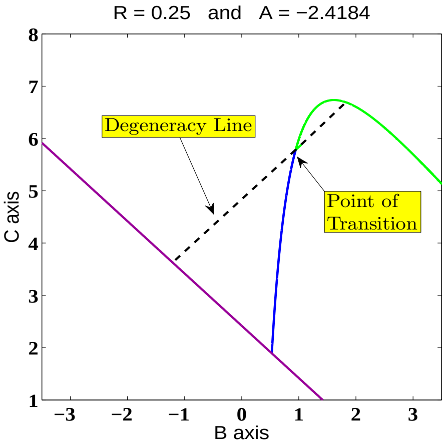

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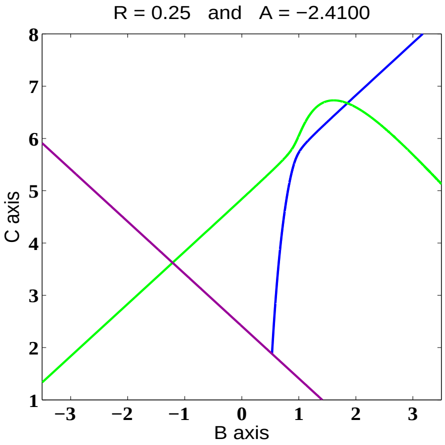
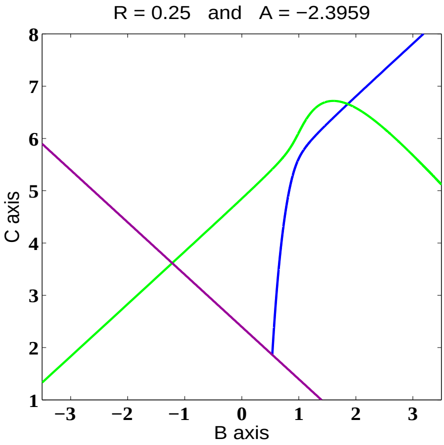
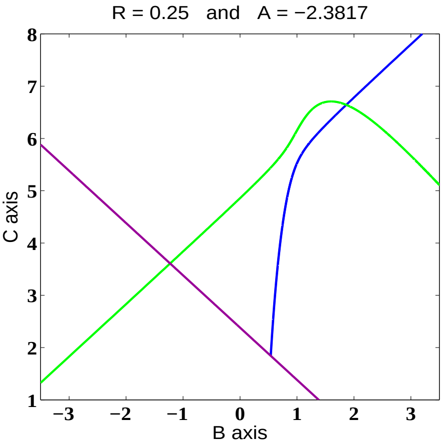


Early Stability Surface



Early Stability Surface



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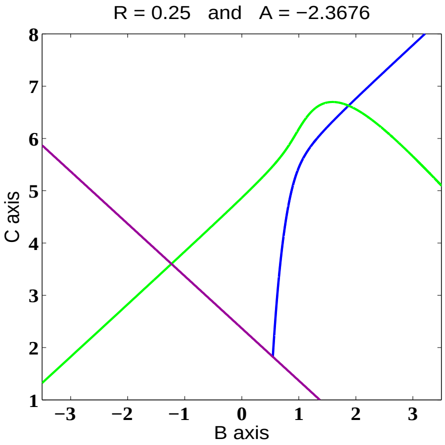

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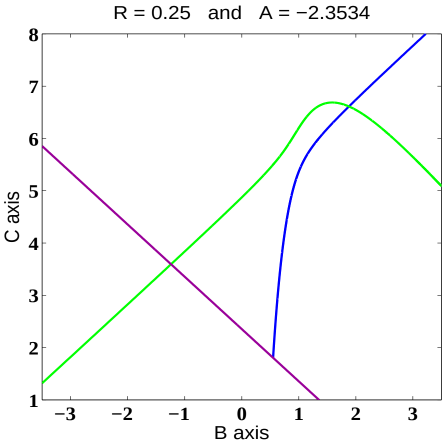

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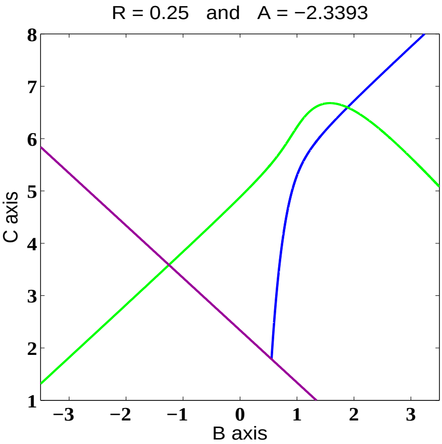

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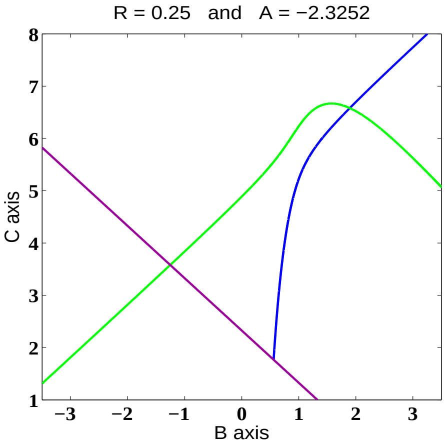

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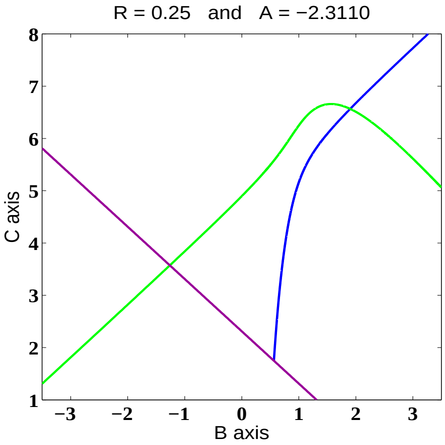

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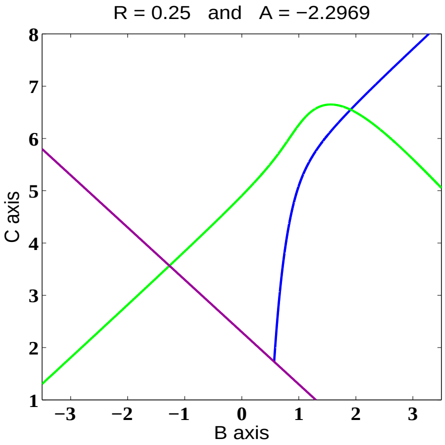

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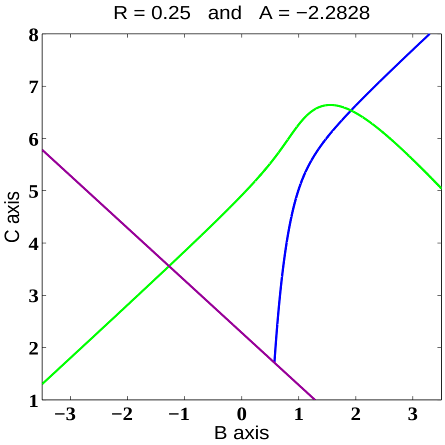

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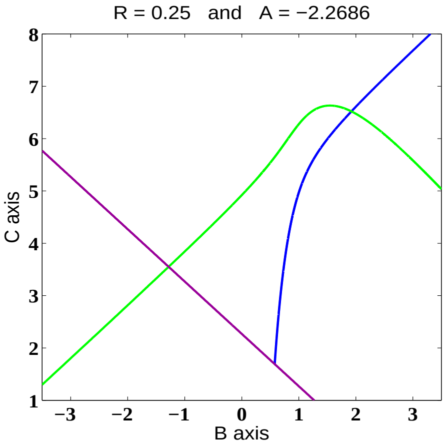

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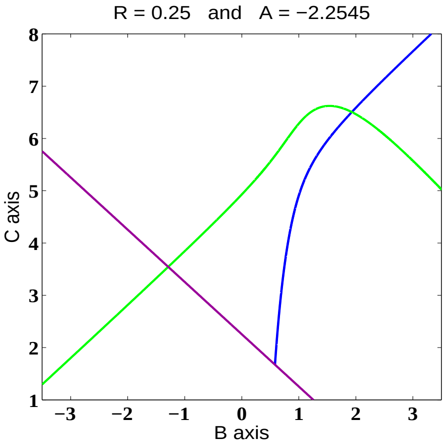

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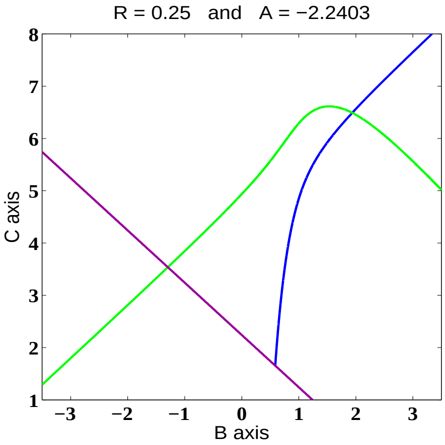

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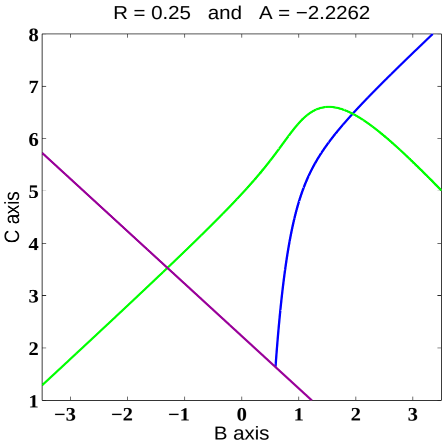

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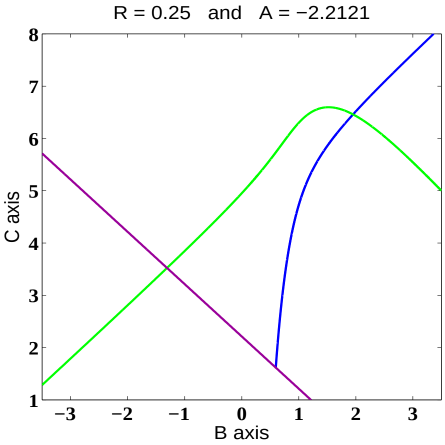

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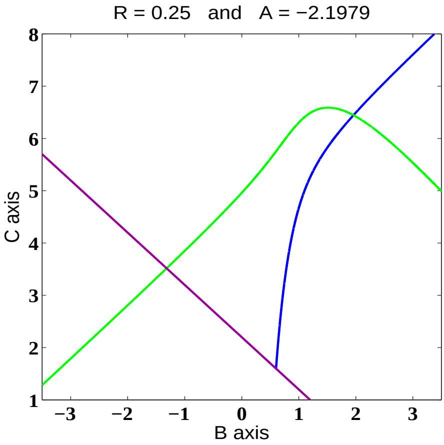

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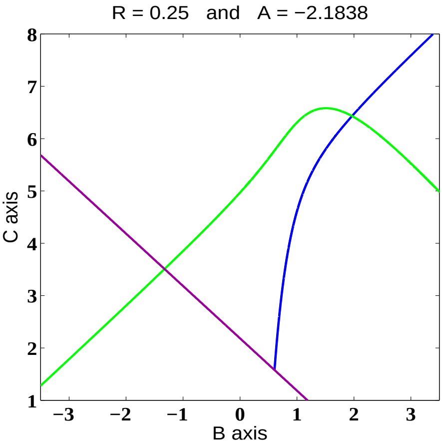

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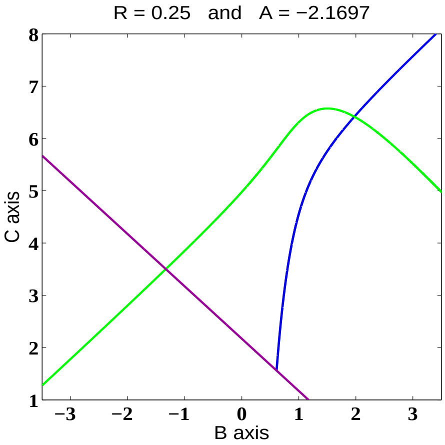

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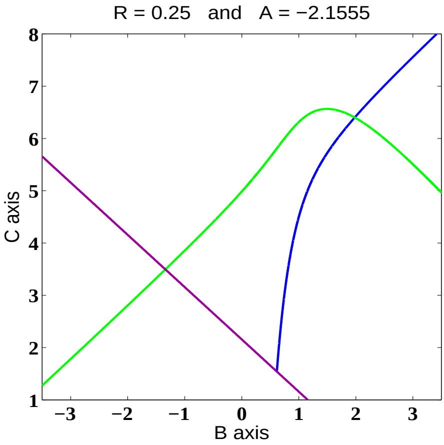

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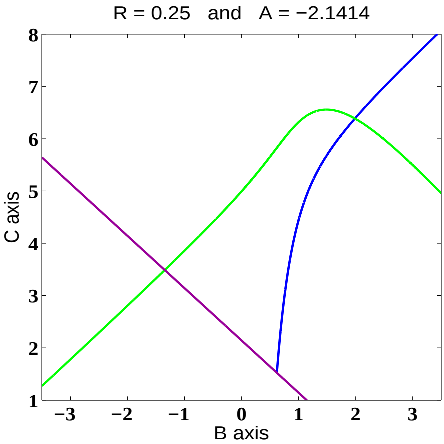

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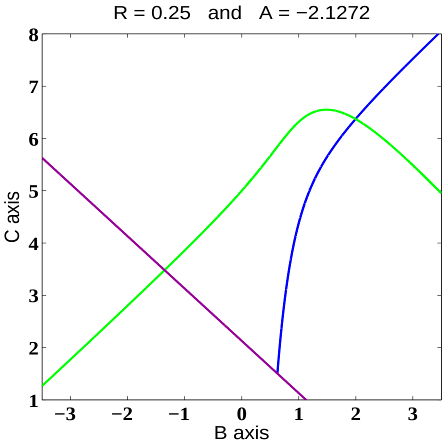

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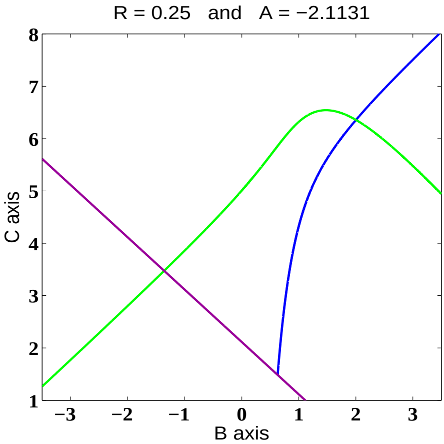

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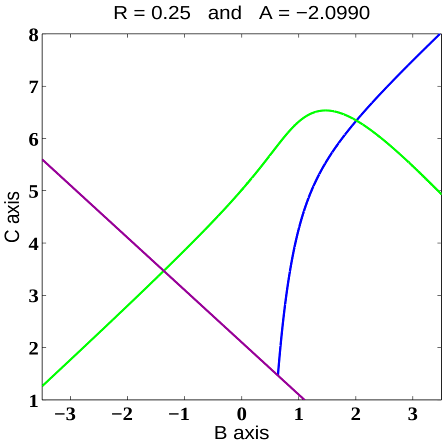

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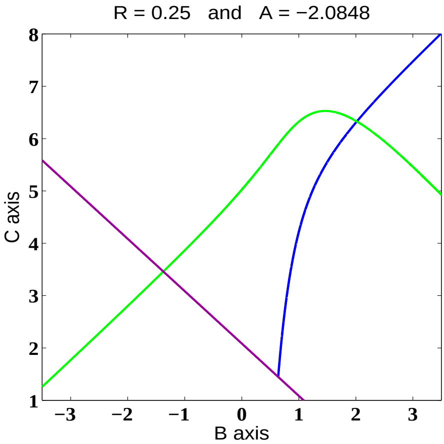

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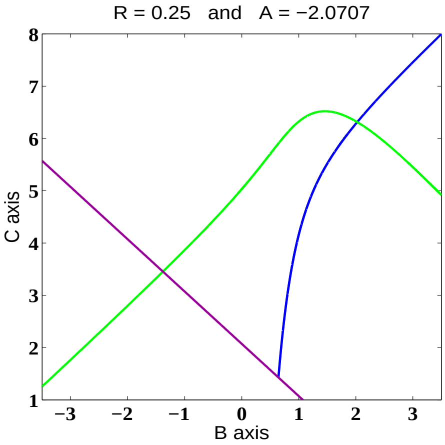

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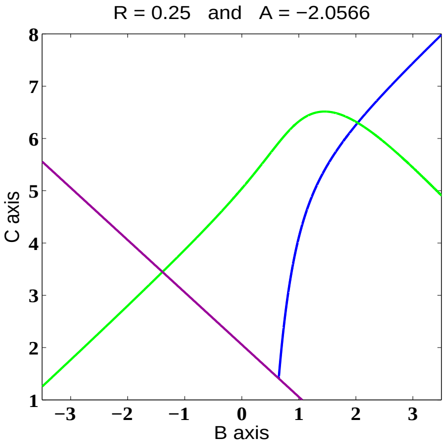

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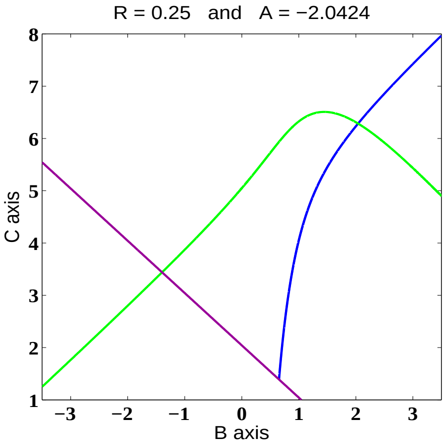

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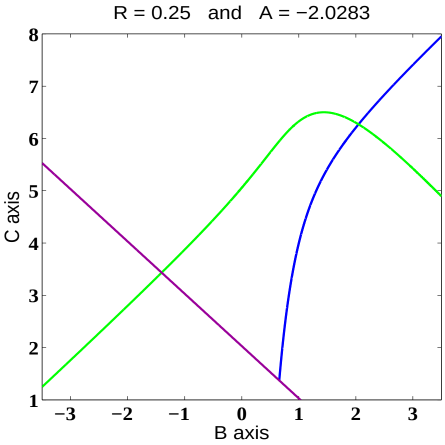

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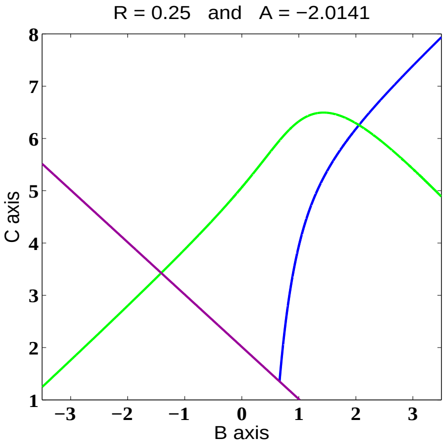

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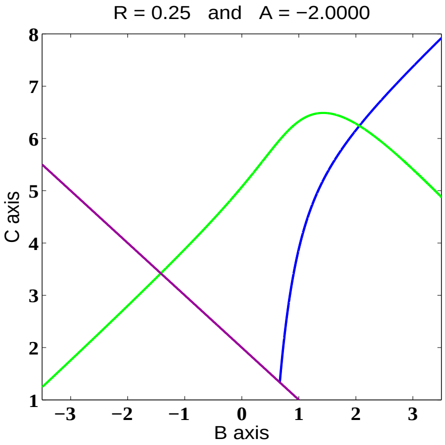
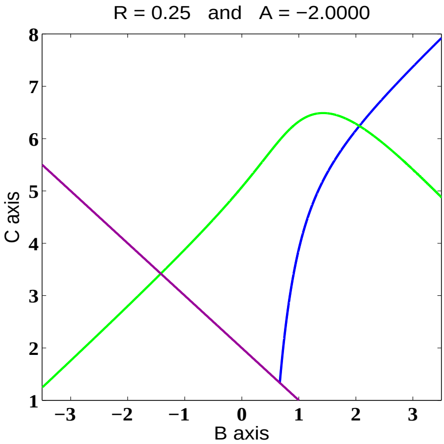
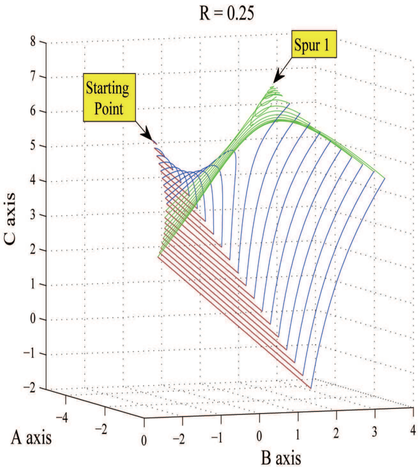


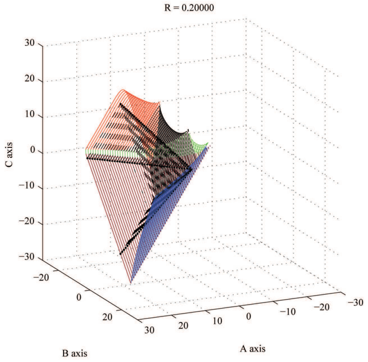
Early Stability Surface



3D View - Early Stability Surface for $R = \frac{1}{4}$



3D View - Early Stability Surface for $R = \frac{1}{5}$



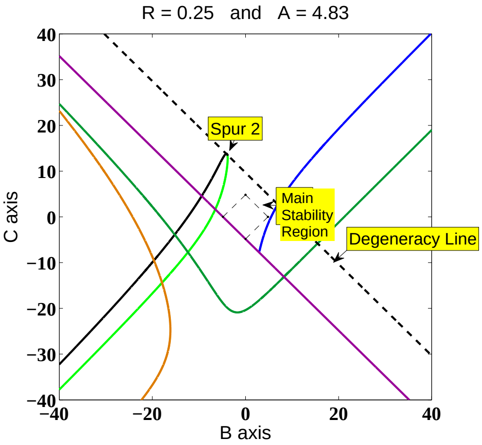
Stability surface comprised of $A \in [-6, 21]$ for $R = \frac{1}{5}$

Transferral and Reverse Transferral

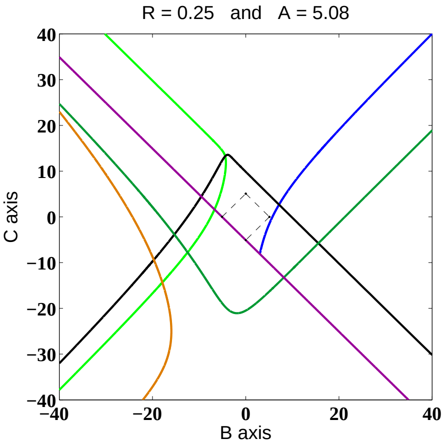
Definition (Transferral and Reverse Transferral)

The **transferral** value of $A = A^z_{i,j}$ is the value of A corresponding to the intersection of Λ_j (or Γ_j) with Λ_i (or Γ_i) at Λ_0. Λ_j (or Γ_j) enters the boundary of the stability region for $A > A^z_{i,j}$. For some values of R the stability surface can undergo a **reverse transferral**, $\tilde{A}^z_{j,i}$, which is a transferral characterized by Λ_j (or Γ_j) leaving the boundary, or a transferring *back over* to Λ_i (or Γ_i) the portion of the boundary originally taken by Λ_j (or Γ_j) at $A^z_{i,j} (< \tilde{A}^z_{j,i})$.

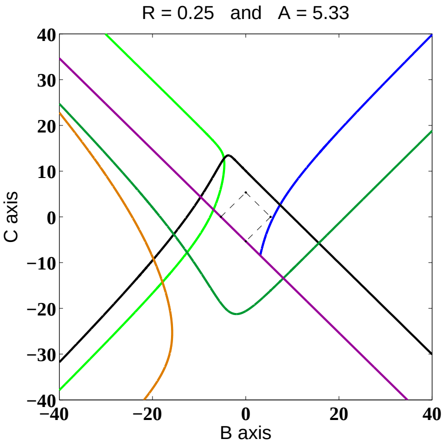
Transferral and Reverse Transferral



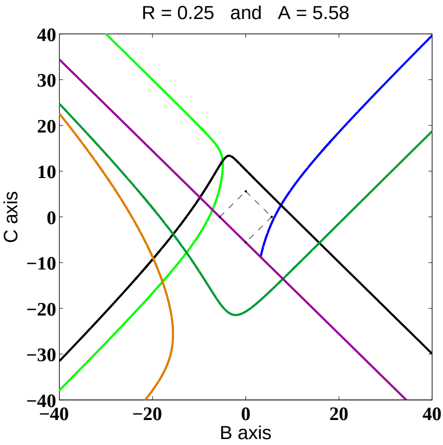
Transferral and Reverse Transferral



Transferral and Reverse Transferral



Transferral and Reverse Transferral



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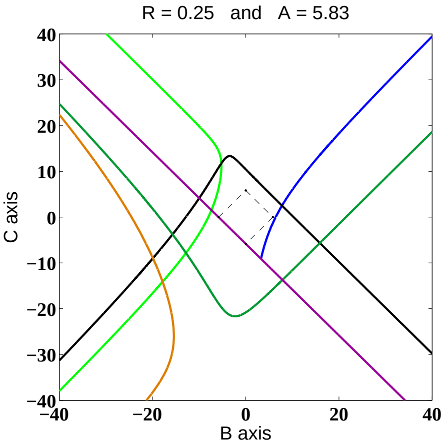

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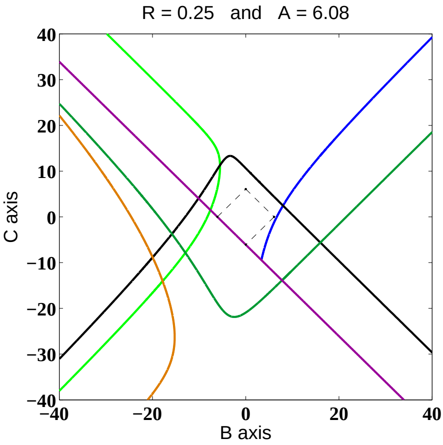

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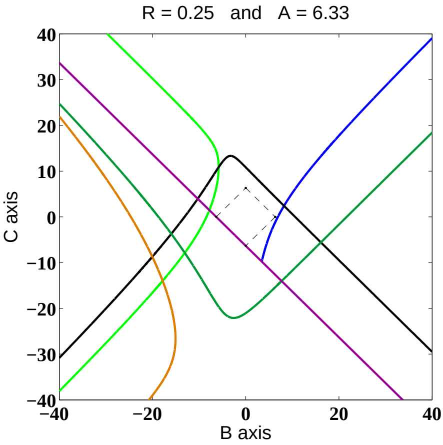

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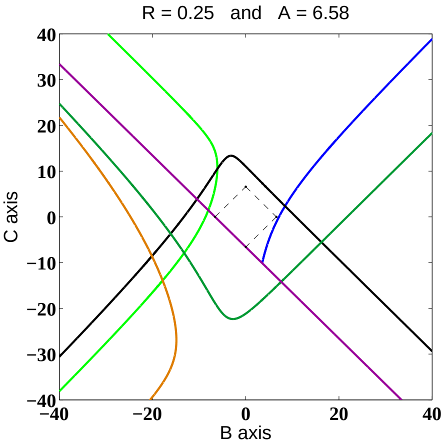

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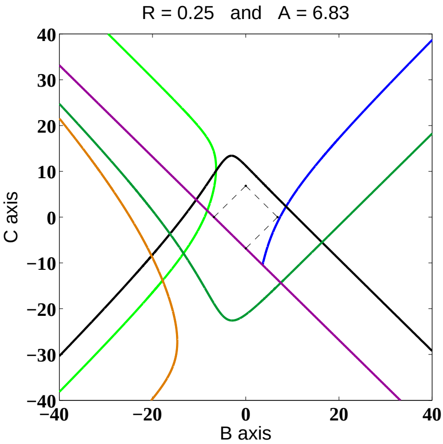

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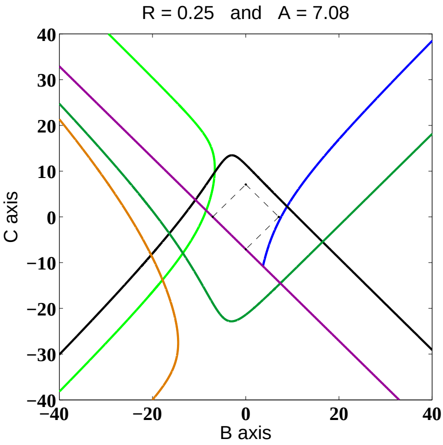
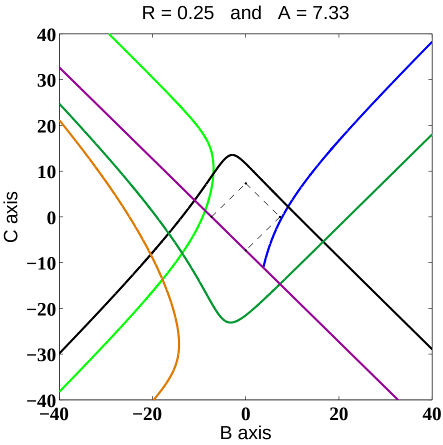


Transferral and Reverse Transferral



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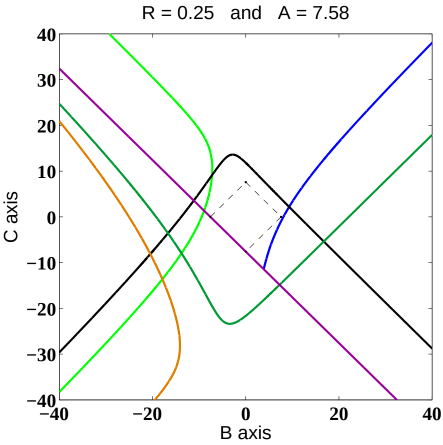

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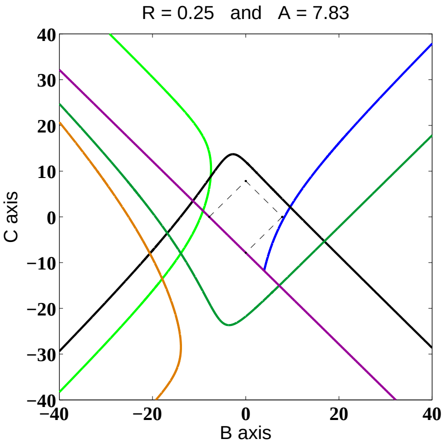

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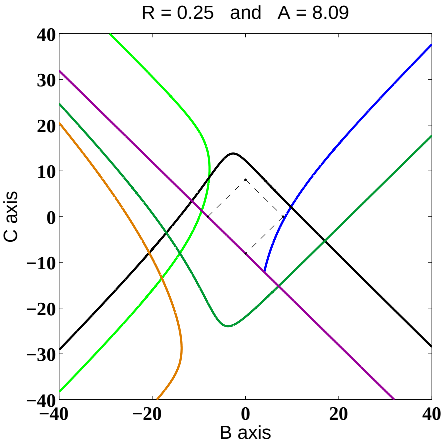

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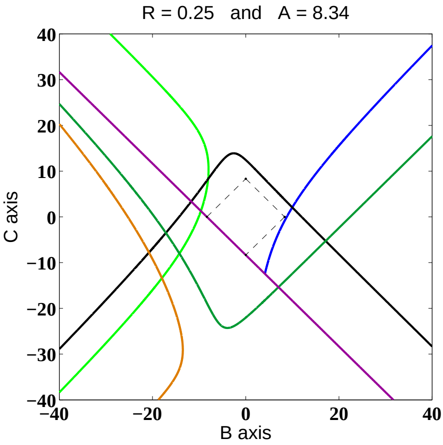
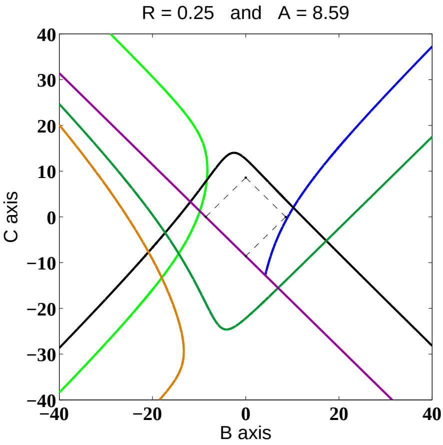


Transferral and Reverse Transferral



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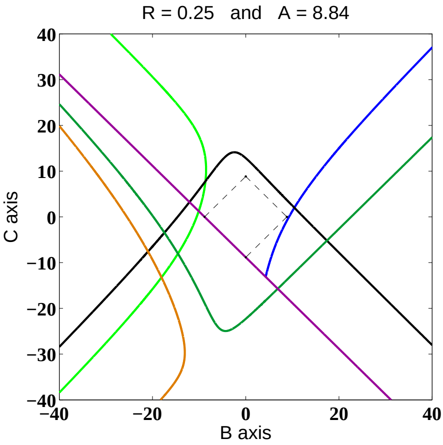

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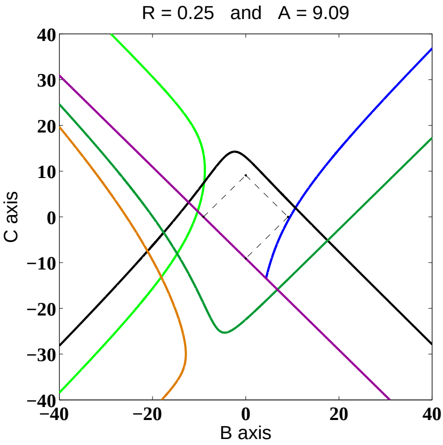

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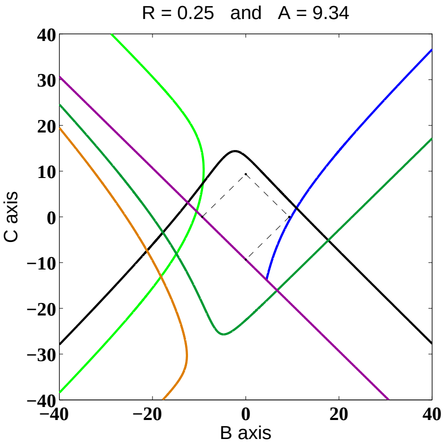
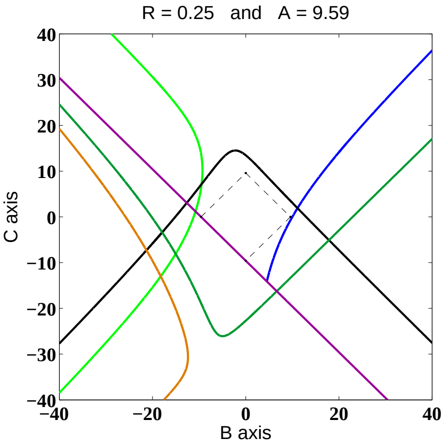
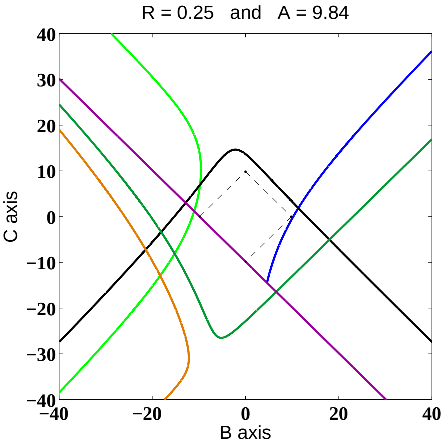


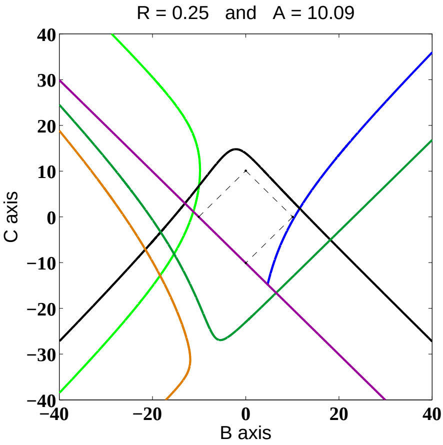
Transferral and Reverse Transferral



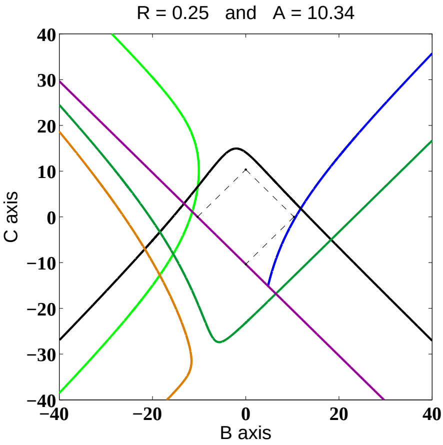
Transferral and Reverse Transferral



Transferral and Reverse Transferral



Transferral and Reverse Transferral



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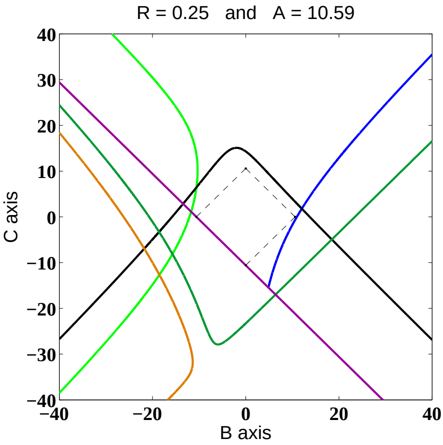

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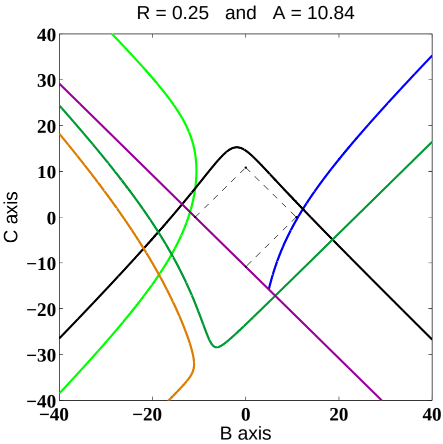

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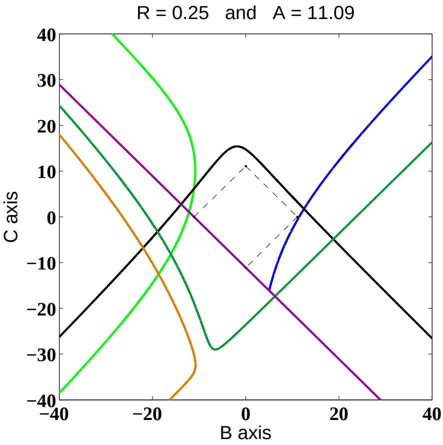

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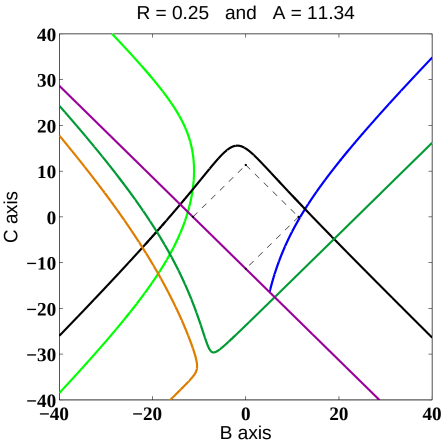

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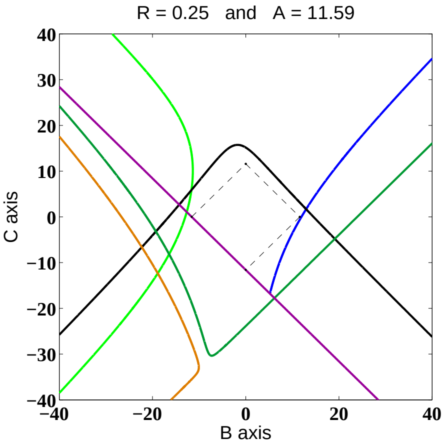

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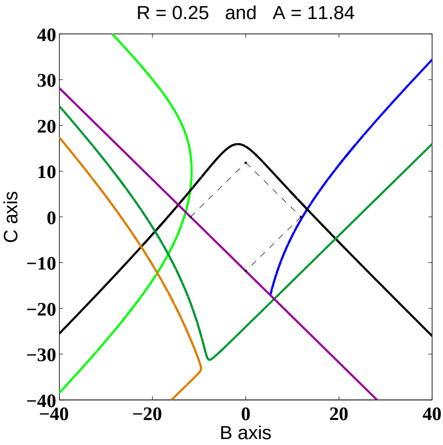

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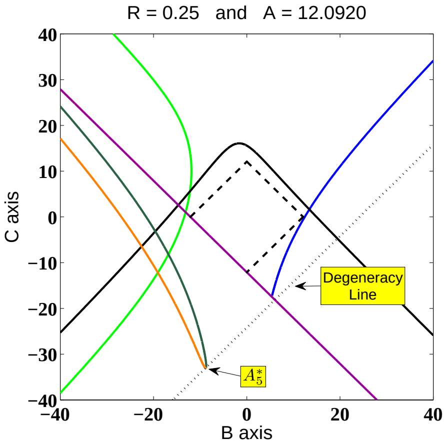

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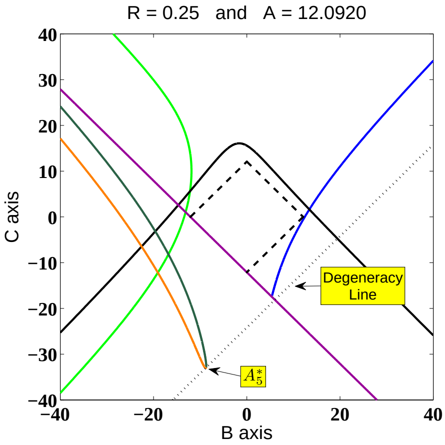

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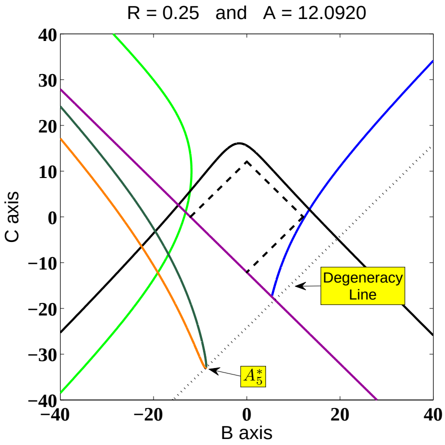

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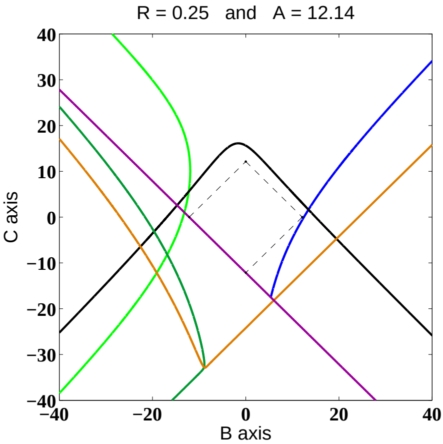

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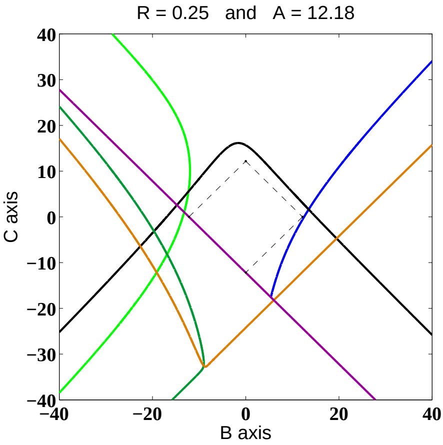

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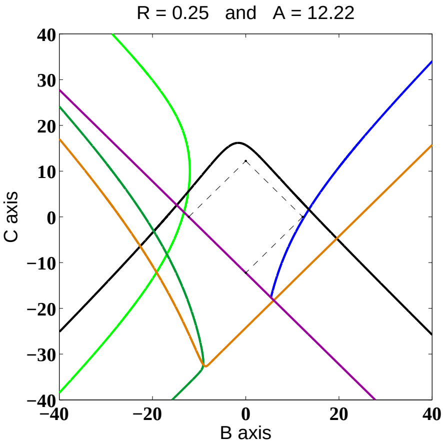

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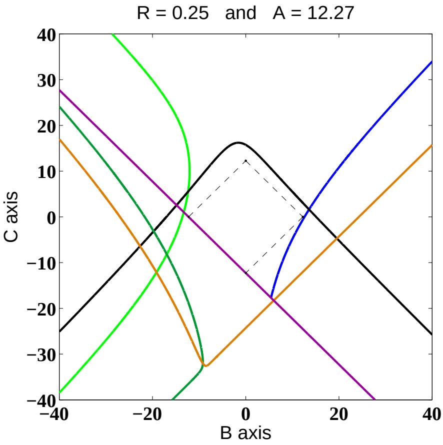

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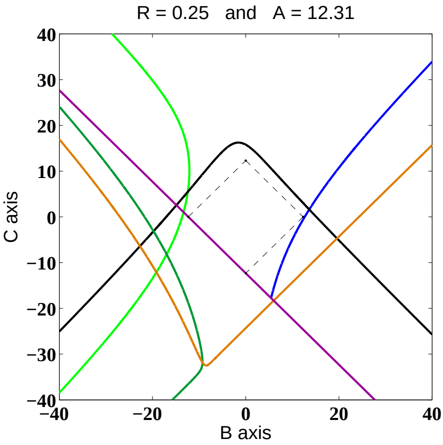
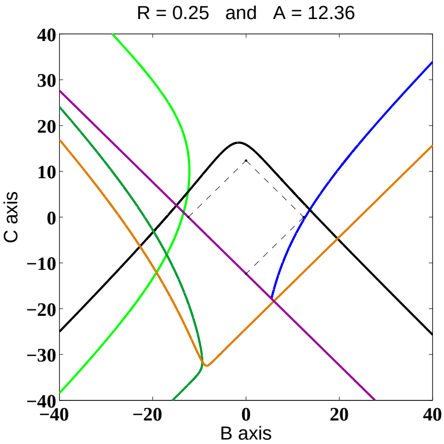


Transferral and Reverse Transferral



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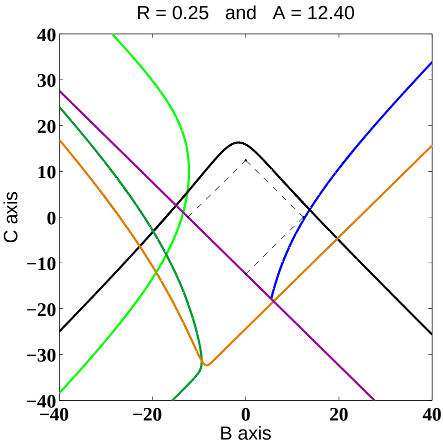

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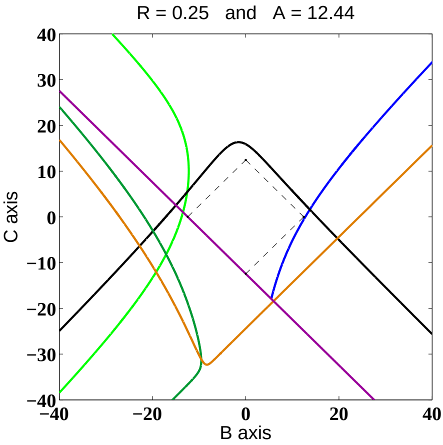

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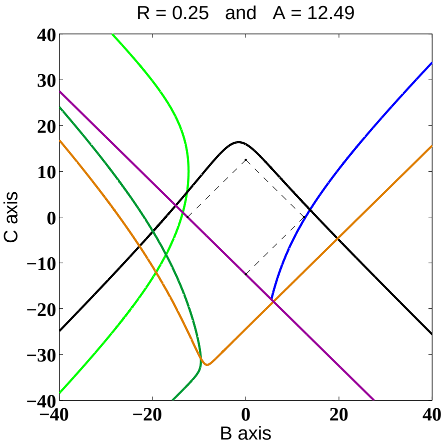

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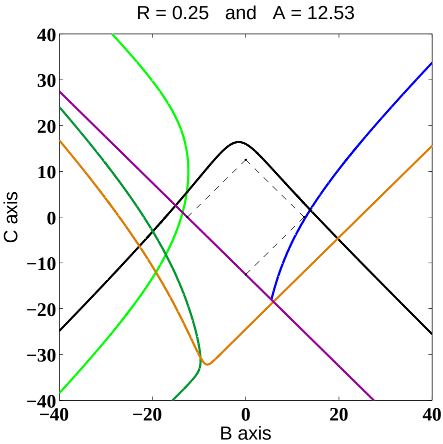

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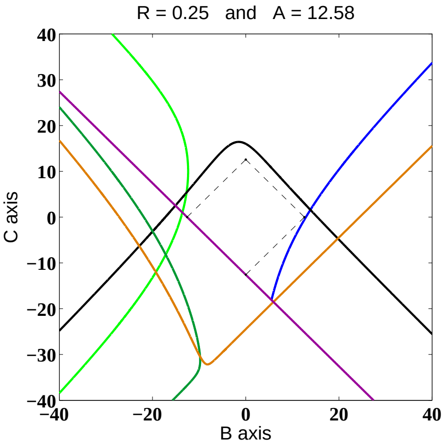

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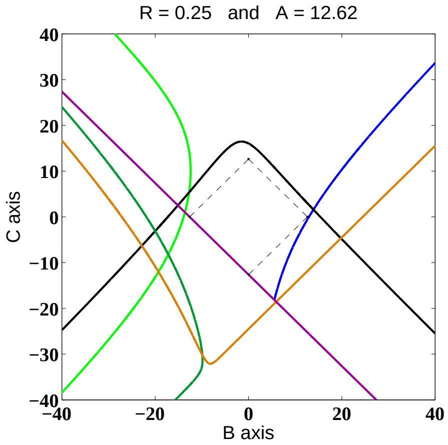

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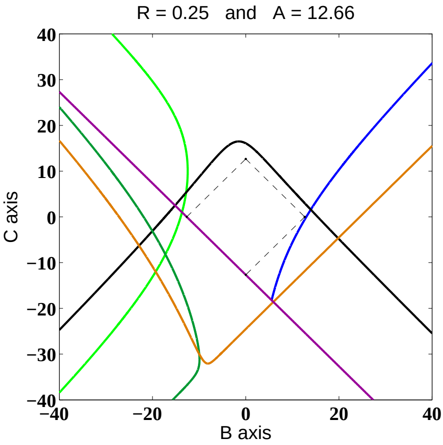

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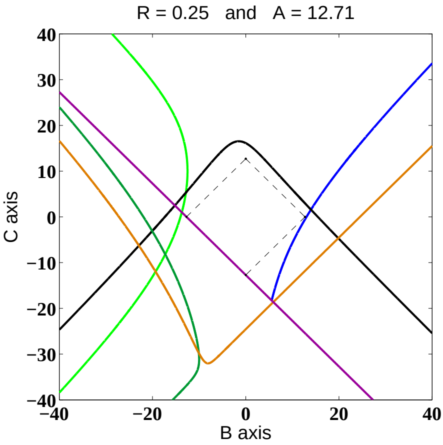

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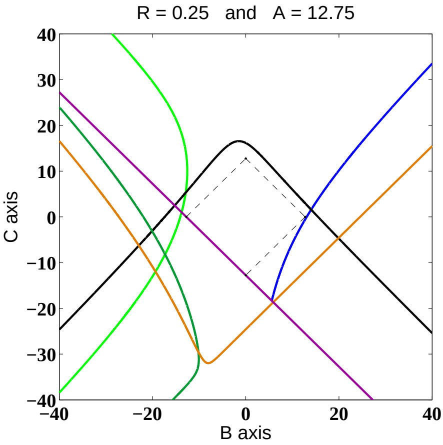

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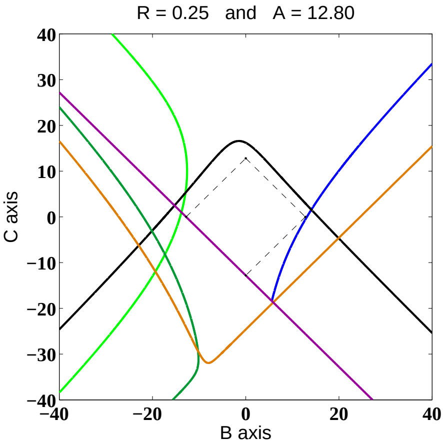

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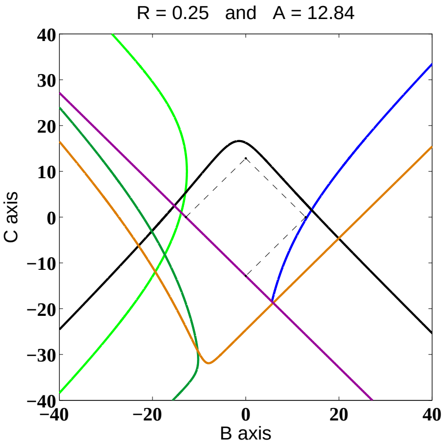

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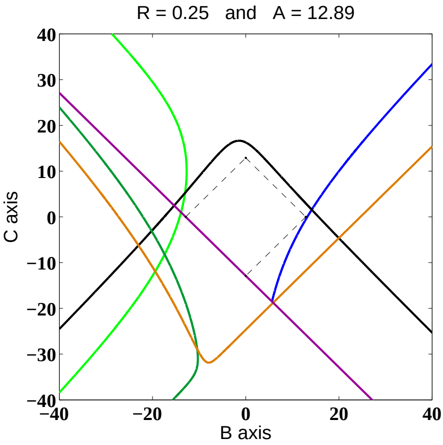

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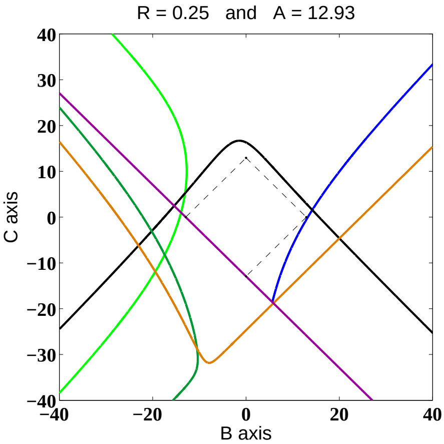

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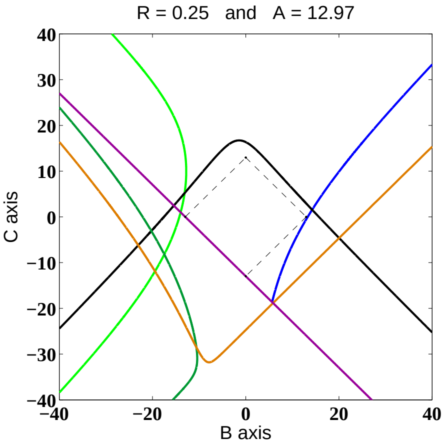

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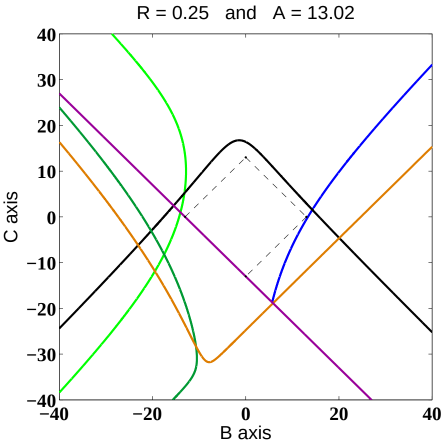

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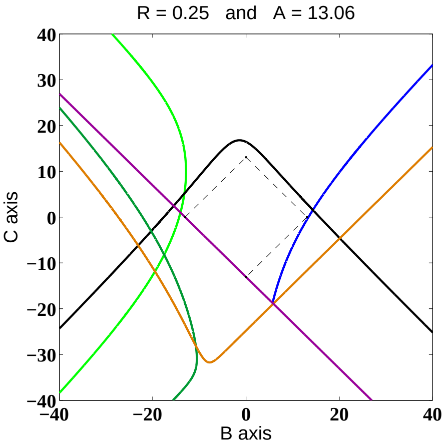

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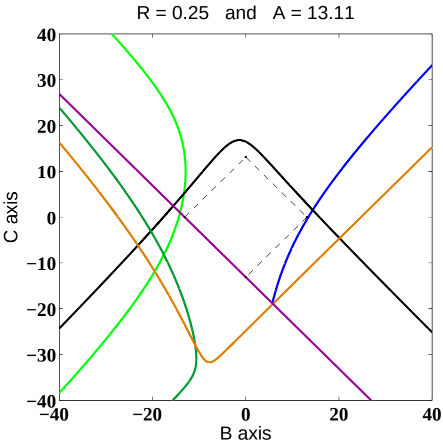

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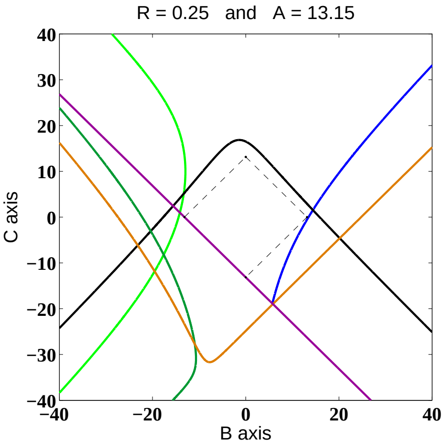

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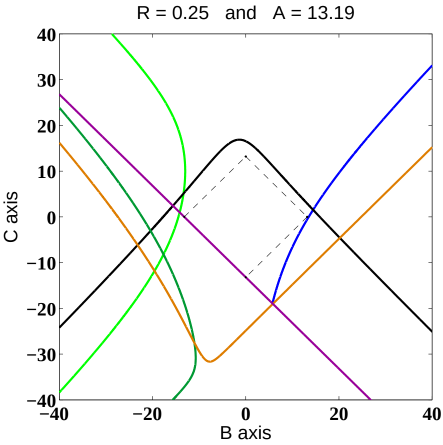

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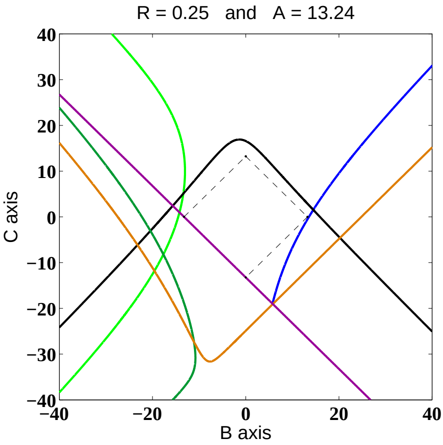

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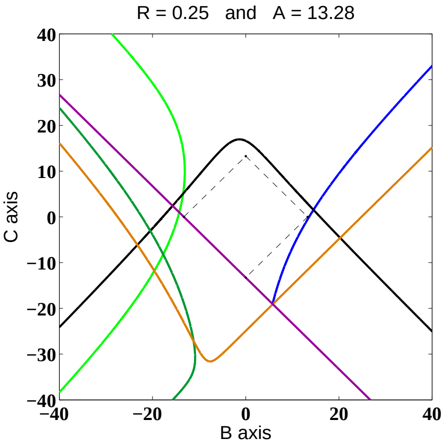

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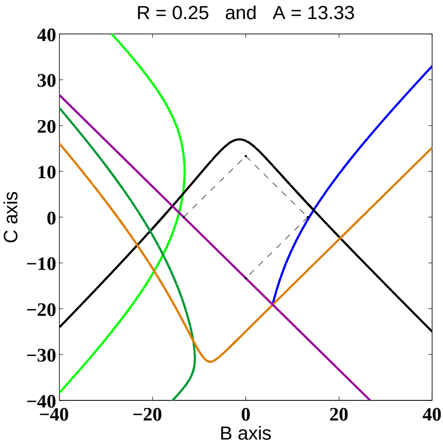

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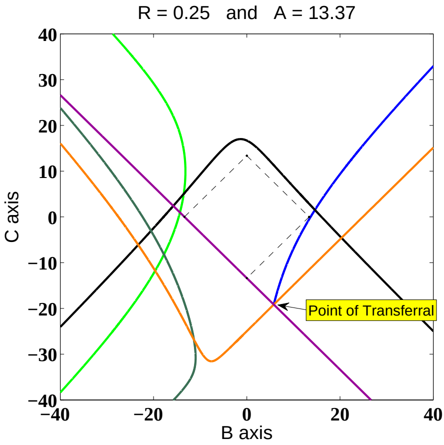

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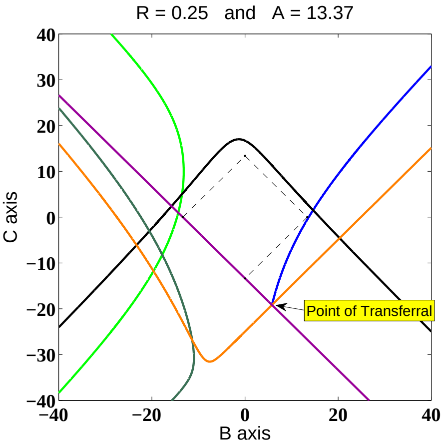
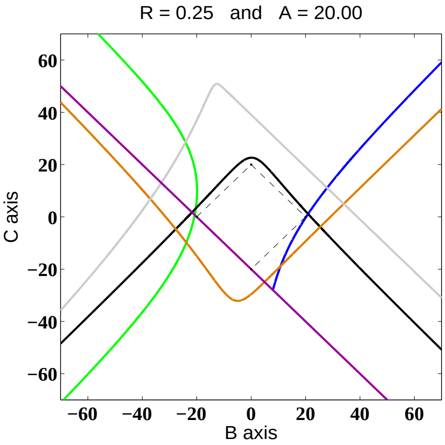


Tangency and Reverse Tangency

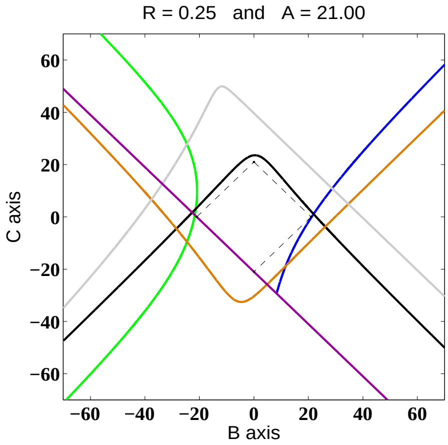
Definition (Tangency and Reverse Tangency)

The value of A corresponding to the **tangency** of two surfaces i and j is denoted $A^t_{i,j}$. Λ_j (or Γ_j) becomes tangent to Λ_i (or Γ_i), where Λ_i (or Γ_i) is a part of the stability boundary prior to $A = A^t_{i,j}$. As A increases from $A^t_{i,j}$, Λ_j (or Γ_j) becomes part of the boundary of the stability region, separating segments of the bifurcation surface to which it was tangent. However, many times as A is increased Λ_j (or Γ_j), the same surface (curve) which entered the boundary through tangency $A^t_{i,j}$, can be seen leaving the stability boundary via a **reverse tangency**, denoted $\tilde{A}^t_{j,i}$.

Tangency and Reverse Tangency



Tangency and Reverse Tangency



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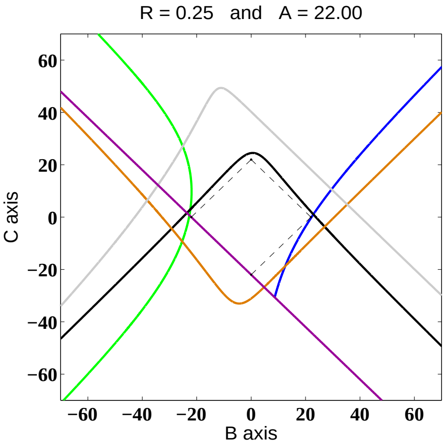

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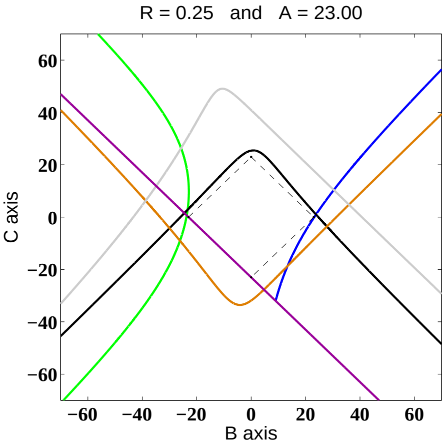

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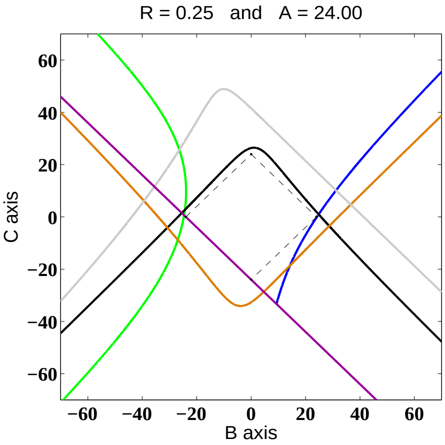

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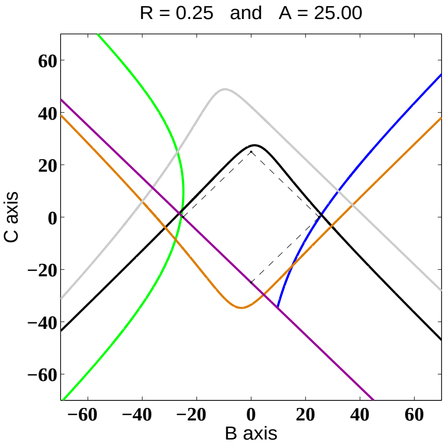

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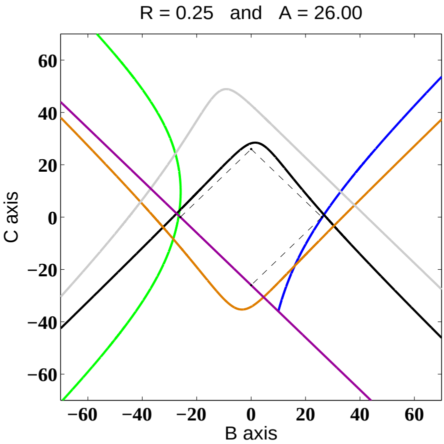

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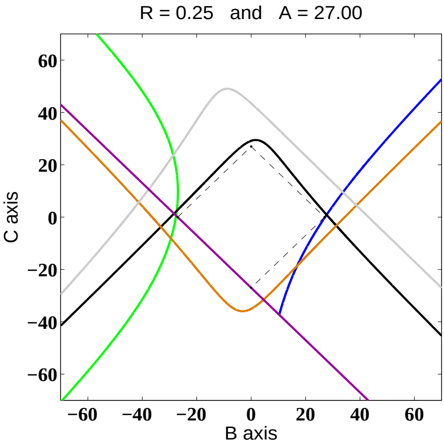

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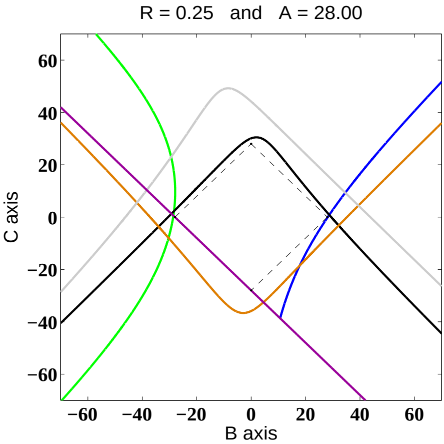

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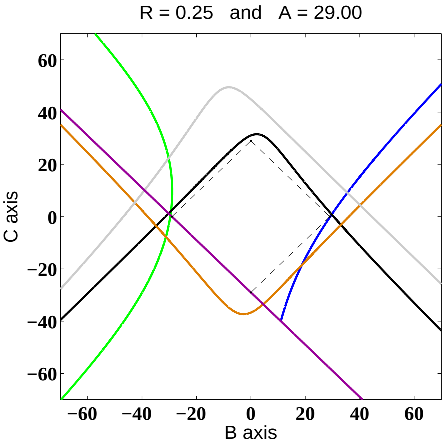

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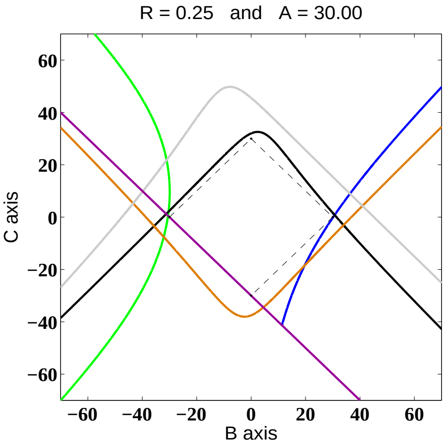

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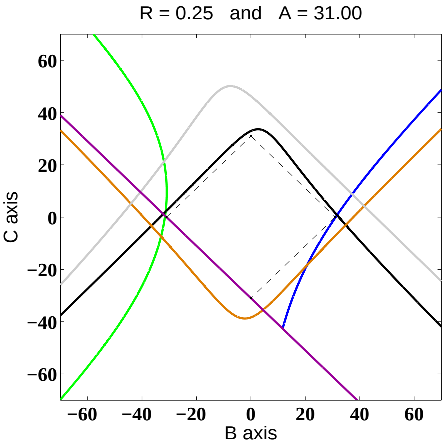

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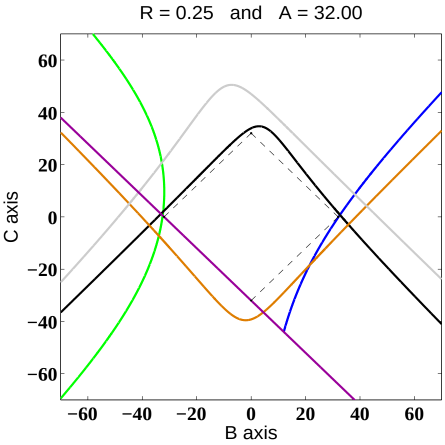

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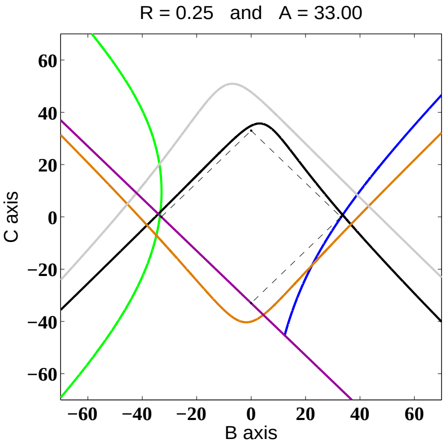

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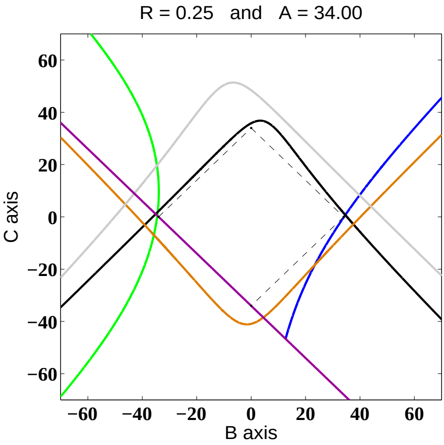

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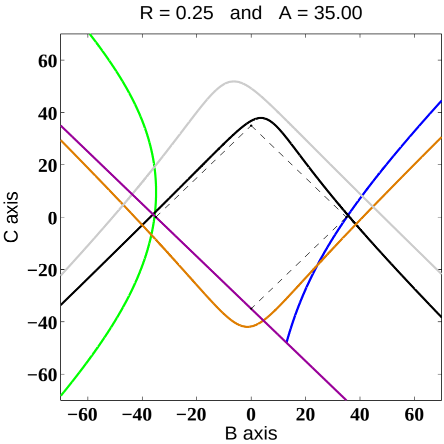

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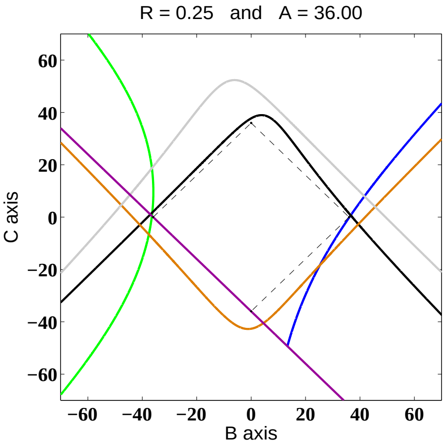

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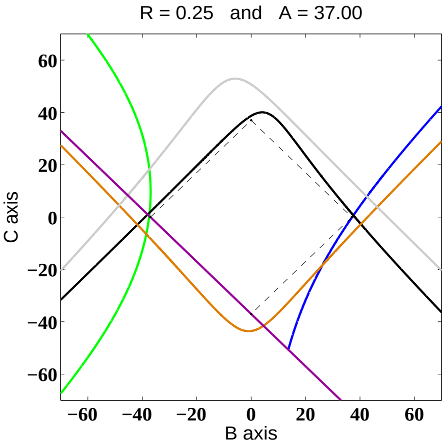

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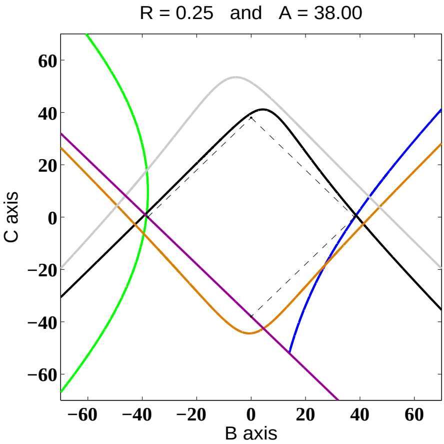

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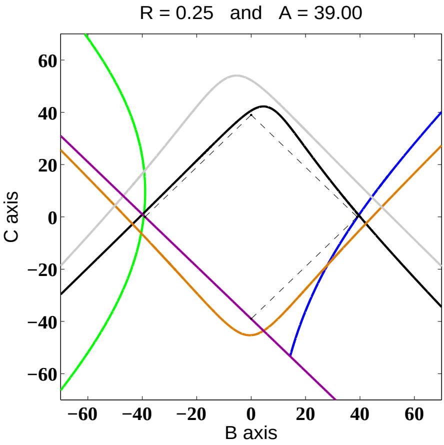

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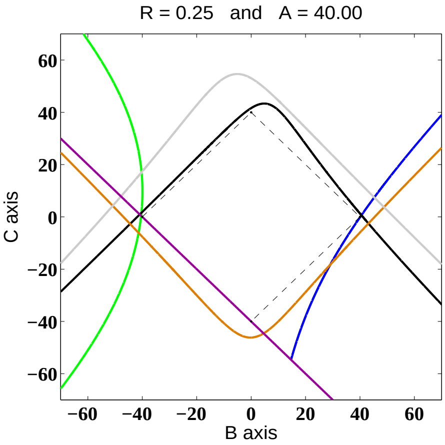

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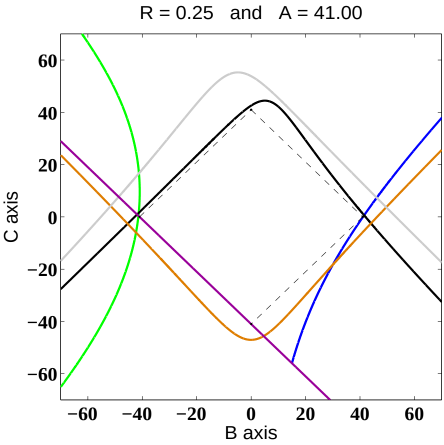

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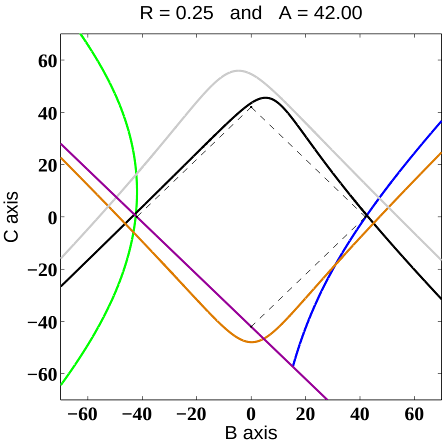

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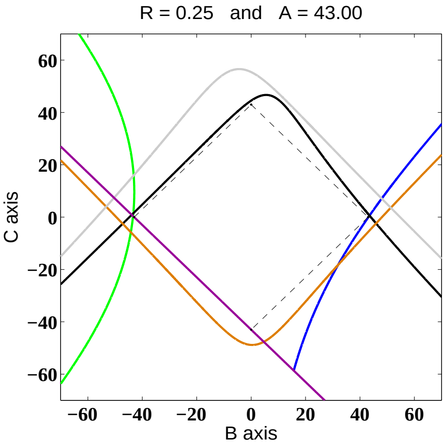

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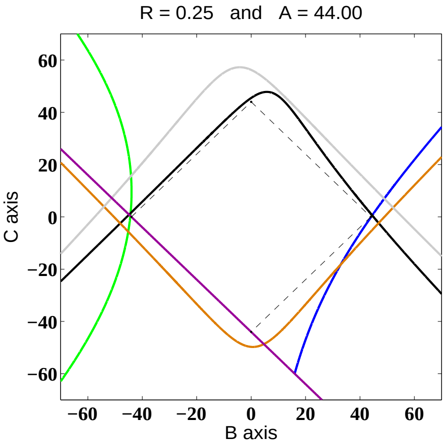

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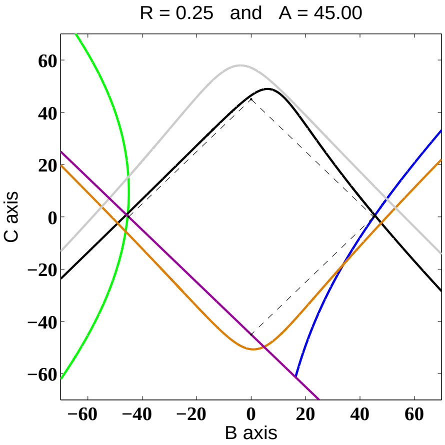

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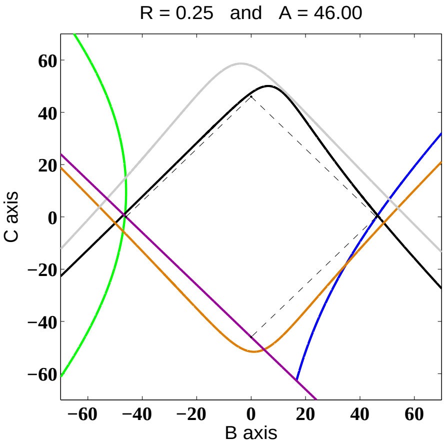

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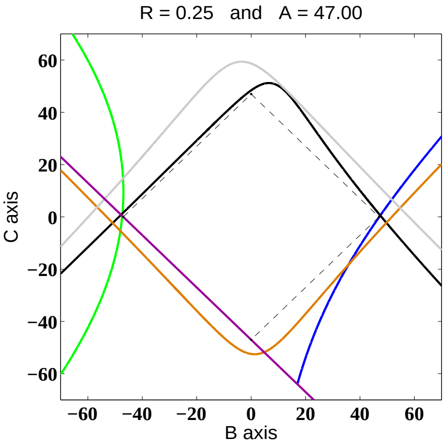

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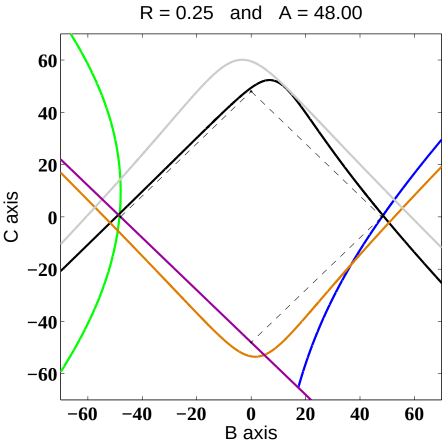

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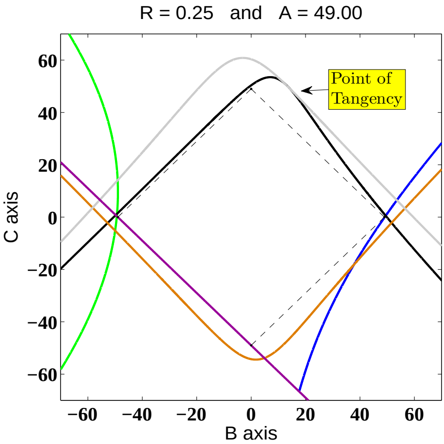

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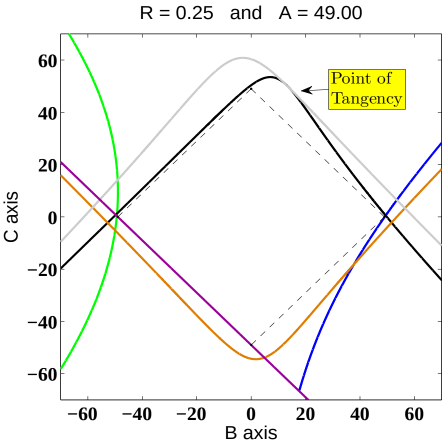

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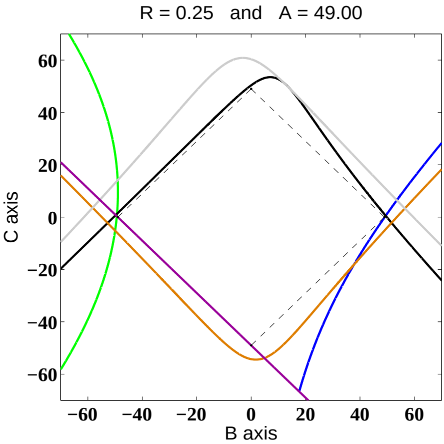

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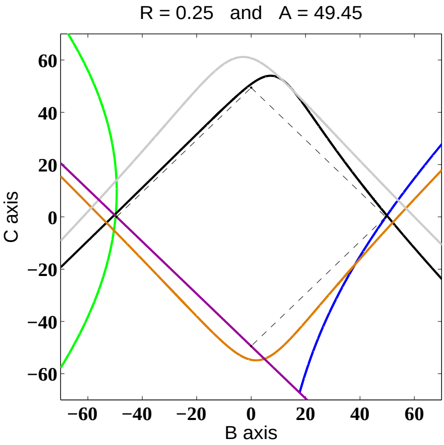

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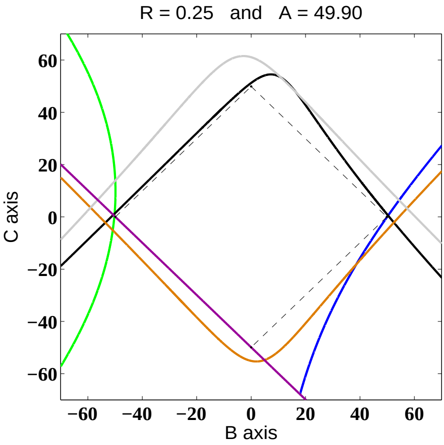

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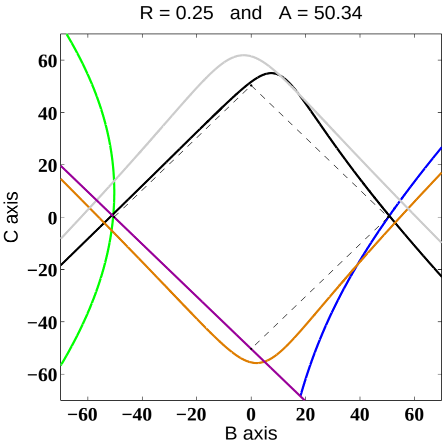

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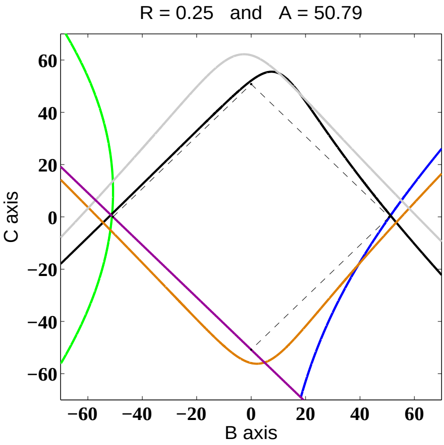

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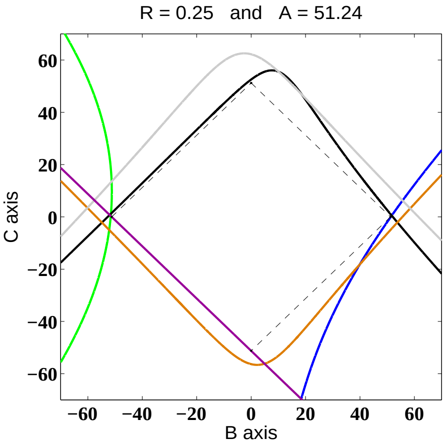

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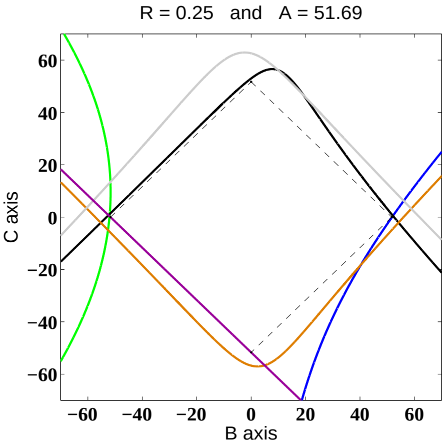

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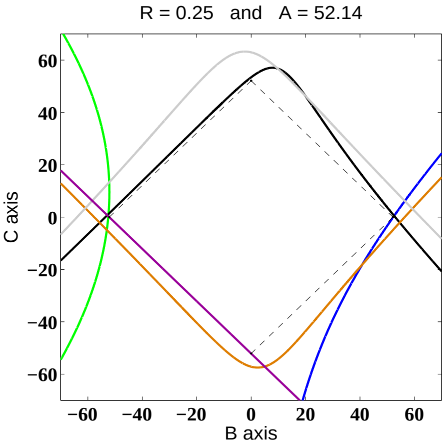

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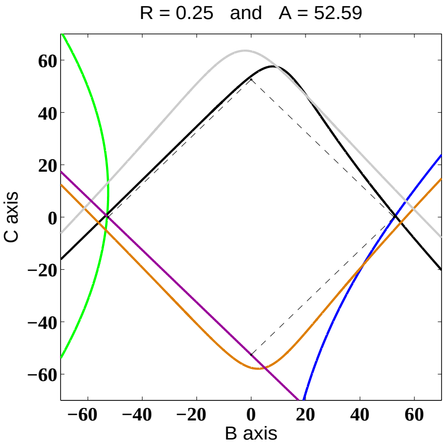

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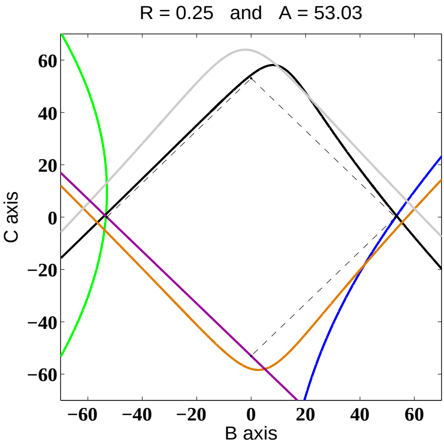

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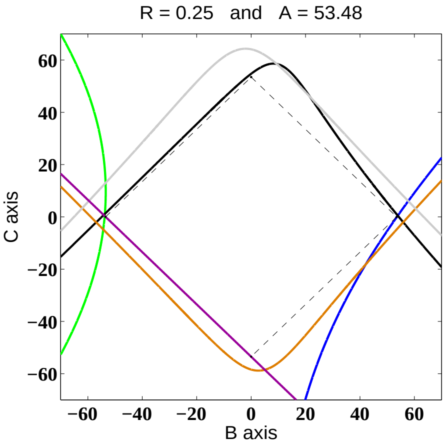

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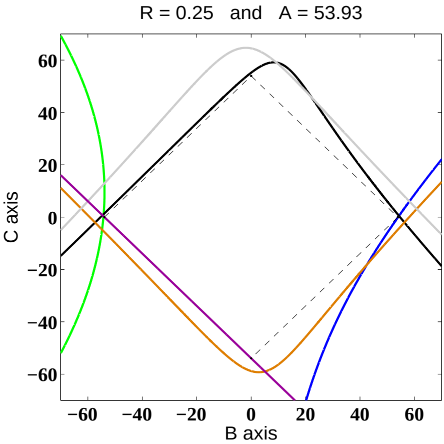

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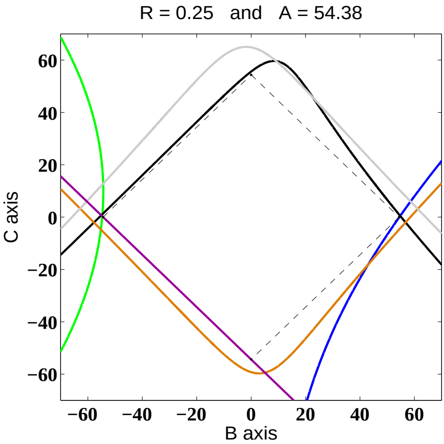
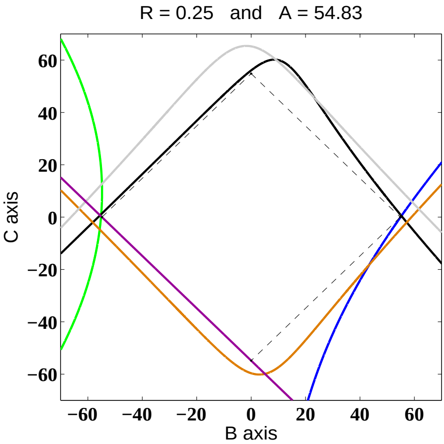
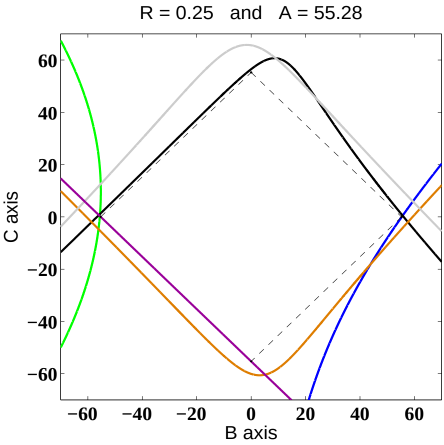


Tangency and Reverse Tangency



Tangency and Reverse Tangency



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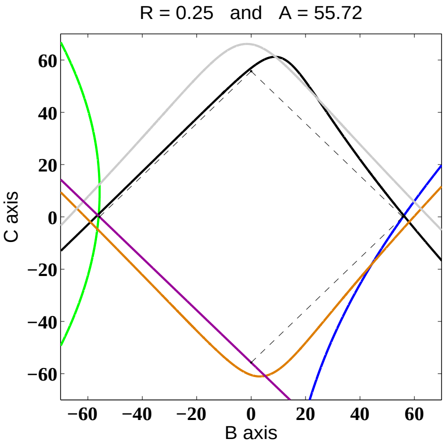

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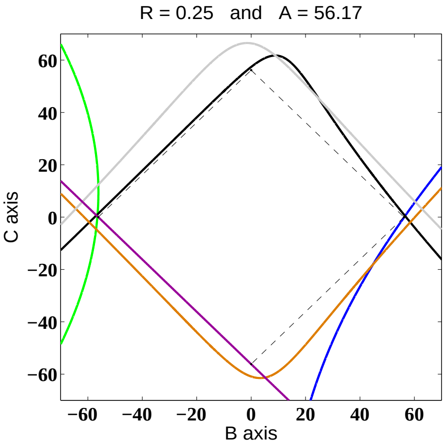

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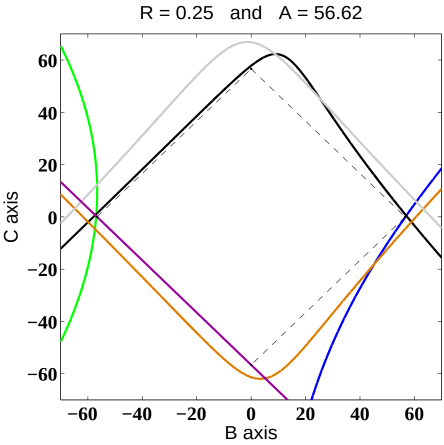

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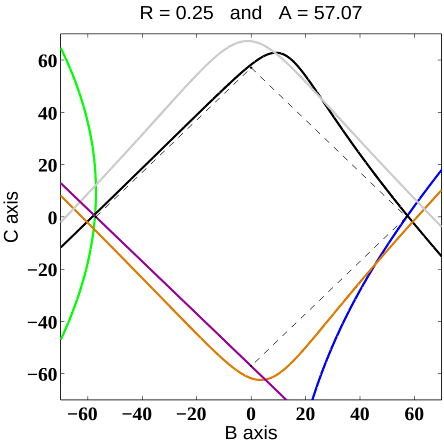
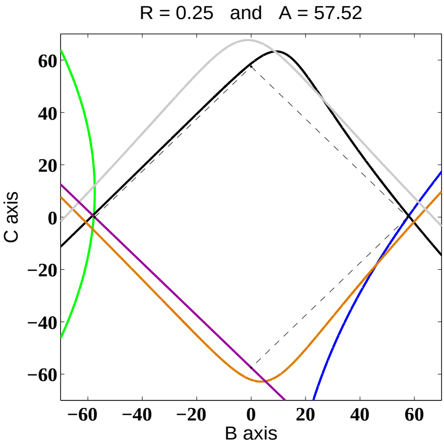
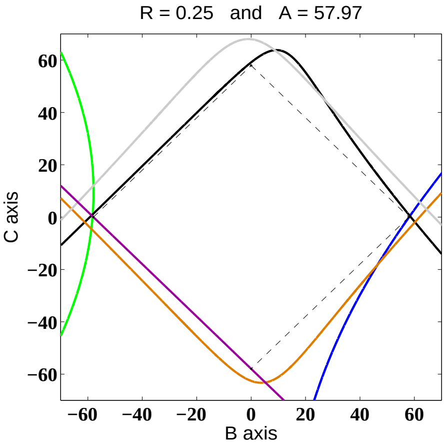


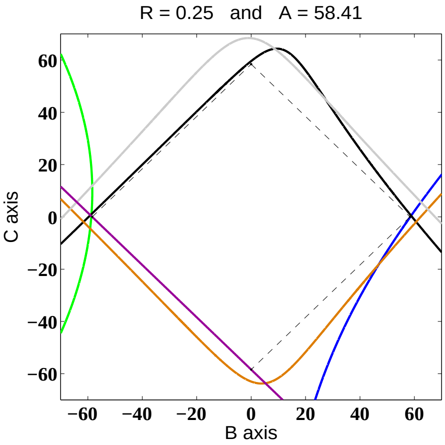
Tangency and Reverse Tangency



Tangency and Reverse Tangency



Tangency and Reverse Tangency



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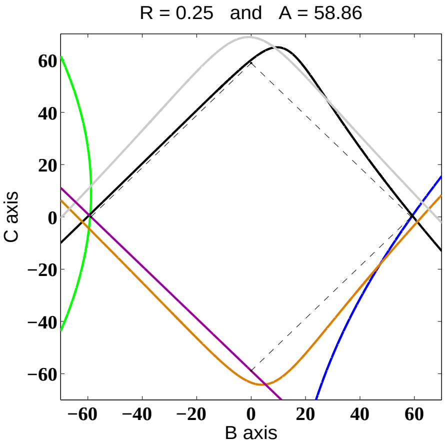
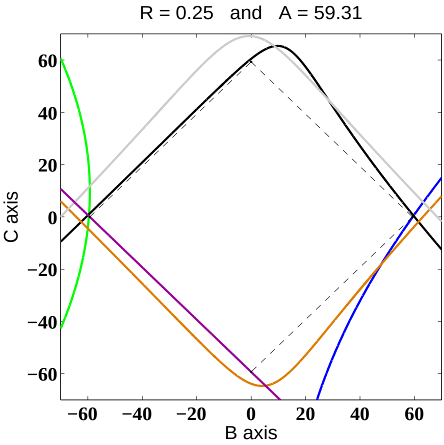
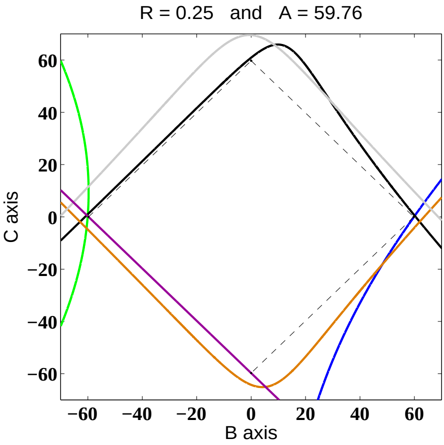


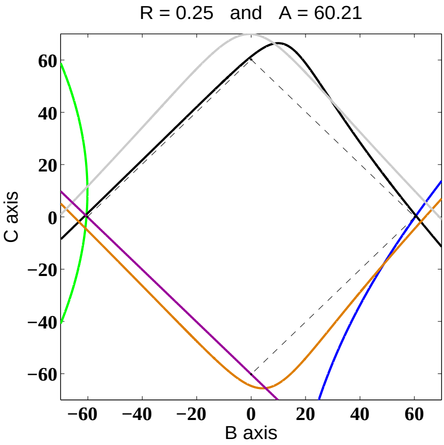
Tangency and Reverse Tangency



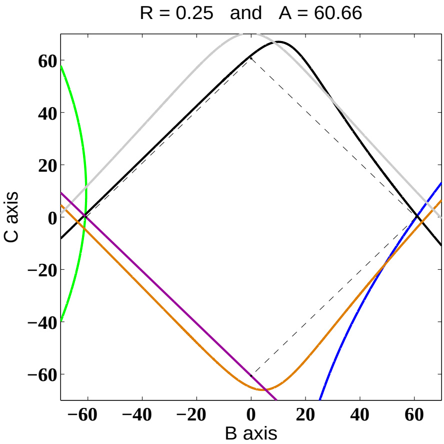
Tangency and Reverse Tangency



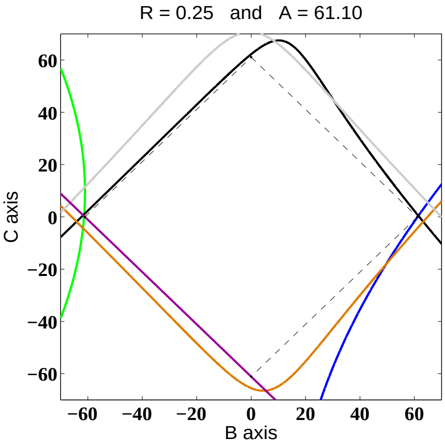
Tangency and Reverse Tangency



Tangency and Reverse Tangency



Tangency and Reverse Tangency



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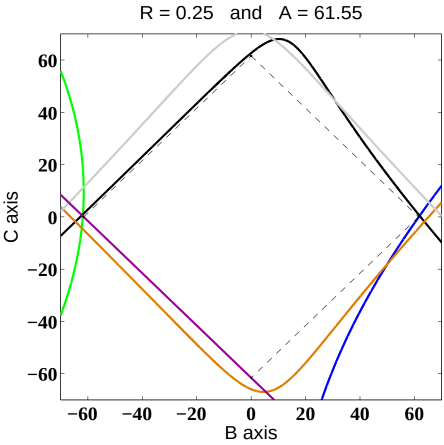
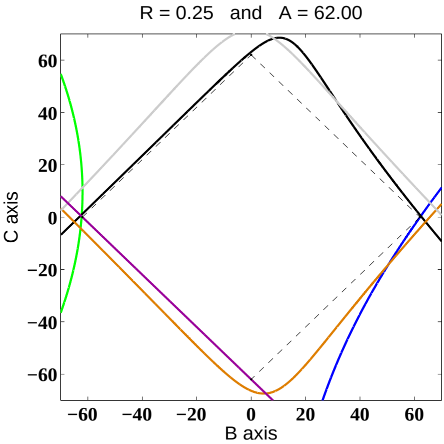
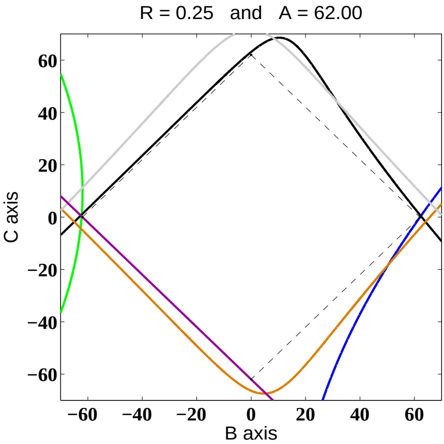


Tangency and Reverse Tangency

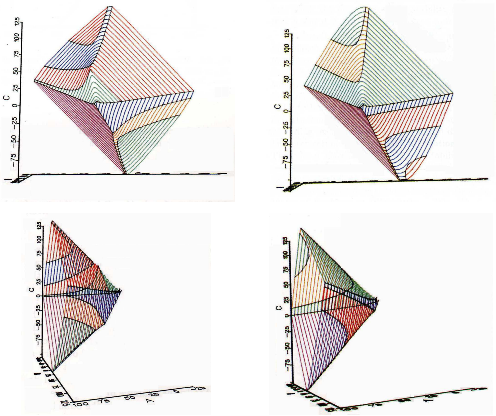


Tangency and Reverse Tangency



Stability Region for $R = 0.31$ and $R = 1/3$

Stability regions for $A \leq 100$ with $R = 0.31$ (left) and $R = 1/3$ (right)



Asymptotic Stability Region for $R = \frac{1}{4}$

- Use definitions to describe stability surface as A increases

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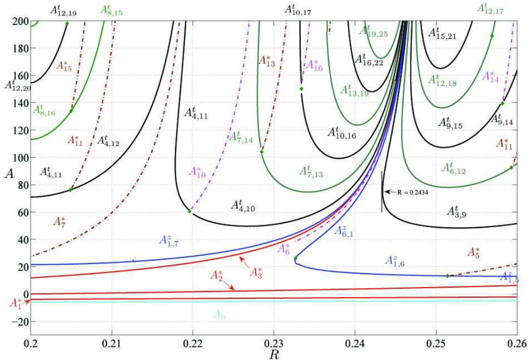
Asymptotic Stability Region for $R = \frac{1}{4}$

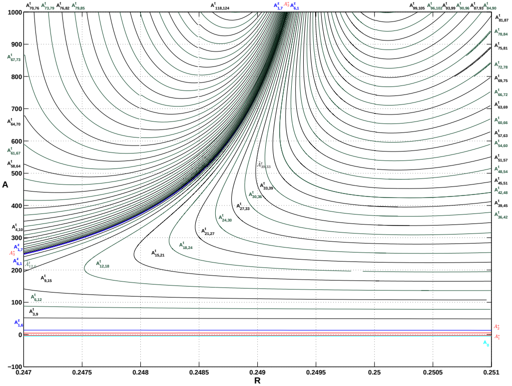
- Use definitions to describe stability surface as A increases
- Show evolution of surface near $R = \frac{1}{4}$
- Detail example of $R = 0.249$
- Appeal to continuity of characteristic equation
- Describe family structure of **bifurcation curves** for $R = \frac{1}{n}$ when n small

Diagram for Transitions, Transferrals, and Tangencies



The A_0, transitions, transferrals, and tangencies for $R \in [0.20, 0.26]$ and $A \leq 200$

Diagram focused near $R = \frac{1}{4}$

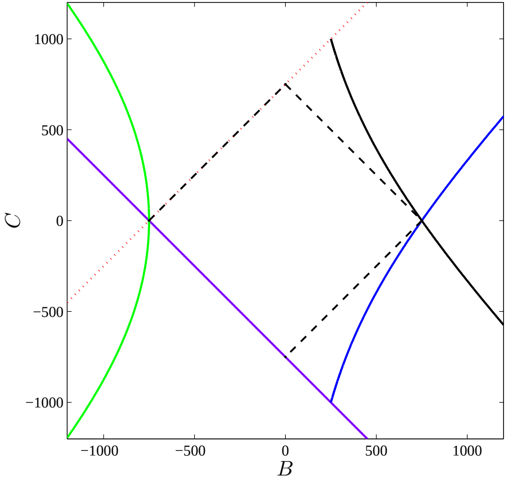


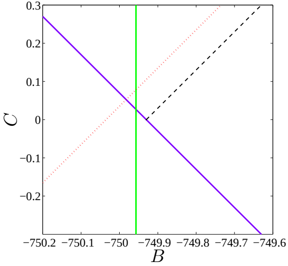
The A_0, transitions, transferrals, and tangencies for $R \in [0.247, 0.251]$ and $A \in [A_0, 1000]$

Table for Changes for $R = 0.249$

A_0	-5.02	tangency	$A^t_{33,39} \approx 462.1$
spur 1	$[A^p_1, A^*_1] \approx [-2.73, -2.45]$	reverse tangency	$\tilde{A}^t_{39,33} \approx 559.2$
spur 2	$[A^p_2, A^*_2] \approx [4.71, 4.71]$	reverse tangency	$\tilde{A}^t_{36,30} \approx 622.3$
transferral	$A^z_{1,6} \approx 13.3$	reverse tangency	$\tilde{A}^t_{33,27} \approx 655.4$
tangency	$A^t_{3,9} \approx 49.4$	reverse tangency	$\tilde{A}^t_{30,24} \approx 678.8$
tangency	$A^t_{6,12} \approx 80.2$	reverse tangency	$\tilde{A}^t_{27,21} \approx 696.7$
tangency	$A^t_{9,15} \approx 108.4$	reverse tangency	$\tilde{A}^t_{24,18} \approx 710.9$
tangency	$A^t_{12,18} \approx 142.5$	reverse tangency	$\tilde{A}^t_{21,15} \approx 722.2$
tangency	$A^t_{15,21} \approx 174.9$	reverse tangency	$\tilde{A}^t_{18,12} \approx 731.2$
tangency	$A^t_{18,24} \approx 208.8$	reverse tangency	$\tilde{A}^t_{15,9} \approx 738.2$
tangency	$A^t_{21,27} \approx 244.7$	reverse tangency	$\tilde{A}^t_{12,6} \approx 743.50$
tangency	$A^t_{24,30} \approx 283.6$	reverse tangency	$\tilde{A}^t_{9,3} \approx 747.1$
tangency	$A^t_{27,33} \approx 327.3$	reverse transferral	$\tilde{A}^z_{6,1} \approx 749.4$
tangency	$A^t_{30,36} \approx 380.0$	spur 3	$A^*_3 \approx 749.93$

Stability Region for $R = 0.249$ at $A_3^* = 749.93$





Five curves on the boundary of the stability region, Γ_0, Γ_1, Γ_2, Γ_3, and Δ_3

Stability Region for $R \to \frac{1}{4}^-$ at A_3^*

- At $A_3^*(R)$ for $R \to \frac{1}{4}^-$, stability region primarily bounded by Γ_0, Γ_1, Γ_3, and Δ_3

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- The intersection of Γ_0 and Γ_1, as well as Γ_3 and Δ_3, extend $\frac{1}{3}$ of the length of a side of the MRS, increasing the stability region
- As $R \to \frac{1}{4}^-$, the stability region at $A_3^*(R)$ is approximately $1.2686\times$Area of MRS

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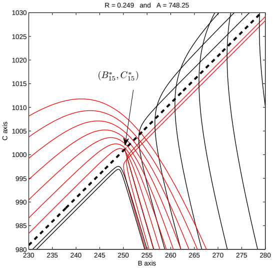
- Showed the typical shape for $R \to \frac{1}{2n}^-$

Process for Reverse Tangency - Transition



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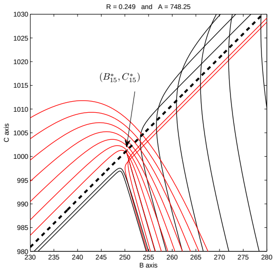
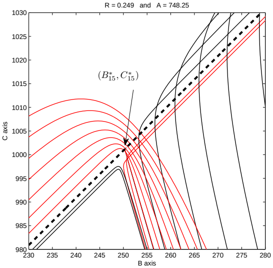


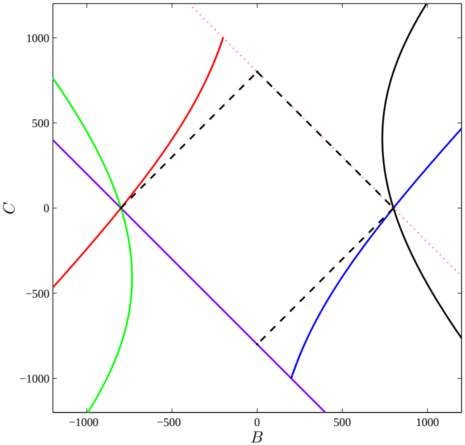
- Showing about 10 bifurcation curves for the 3^{rd} and 4^{th} families with Δ_{15} for $R = 0.249$ at $A^*_{15} = 748.25$

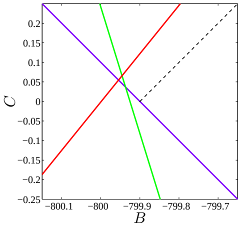
Process for Reverse Tangency - Transition

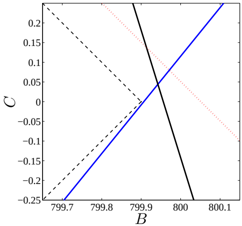


- Showing about 10 bifurcation curves for the 3^{rd} and 4^{th} families with Δ_{15} for $R = 0.249$ at $A^*_{15} = 748.25$
- Γ_3 and Γ_9 remain close to the boundary of the stability region
- $\tilde{A}^t_{9,3} \approx 747.134$ has recently occurred, removing Γ_9 from the boundary of the stability region

Stability Region for $R = 0.199$ at $A_4^* = 799.9$







Six curves on the boundary of the stability region, Γ_0, Γ_1, Γ_4, and Δ_4 with small segments of Γ_2 and Γ_3

Stability Region for $R \to \frac{1}{5}^-$ at A_4^*

- At $A_4^*(R)$ for $R \to \frac{1}{5}^-$, stability region primarily bounded by Γ_0, Γ_1, Γ_4, and Δ_4

Stability Region for $R \to \frac{1}{5}^{-}$ at A_4^*

- At $A_4^*(R)$ for $R \to \frac{1}{5}^{-}$, stability region primarily bounded by Γ_0, Γ_1, Γ_4, and Δ_4
- $\Delta_4 \to$ MRS as $R \to \frac{1}{5}^{-}$

Stability Region for $R \to \frac{1}{5}^-$ at A_4^*

- At $A_4^*(R)$ for $R \to \frac{1}{5}^-$, stability region primarily bounded by Γ_0, Γ_1, Γ_4, and Δ_4
- $\Delta_4 \to$ MRS as $R \to \frac{1}{5}^-$
- The intersection of Γ_0 and Γ_1, as well as Γ_4 and Δ_4, extend $\frac{1}{4}$ of the length of a side of the MRS, increasing the stability region
- As $R \to \frac{1}{5}^-$, the stability region at $A_4^*(R)$ is approximately $1.1859 \times$ Area of MRS

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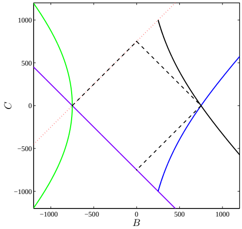
- $\Delta_4 \to$ MRS as $R \to \frac{1}{5}^-$
- The intersection of Γ_0 and Γ_1, as well as Γ_4 and Δ_4, extend $\frac{1}{4}$ of the length of a side of the MRS, increasing the stability region
- As $R \to \frac{1}{5}^-$, the stability region at $A_4^*(R)$ is approximately $1.1859 \times$Area of MRS
- Showed the typical shape for $R \to \frac{1}{2n+1}^-$

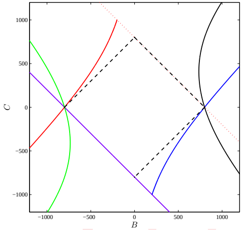
Area Increase of Stability Region for various R

R	Area Ratio	Linear Extension
$\frac{1}{2}$	2.0000	1.0000
$\frac{1}{3}$	1.4431	0.5000
$\frac{1}{4}$	1.2686	0.3333
$\frac{1}{5}$	1.1859	0.2500
$\frac{1}{6}$	1.1386	0.2000
$\frac{1}{7}$	1.1084	0.1667
$\frac{1}{8}$	1.0878	0.1429
$\frac{1}{9}$	1.0729	0.1250
$\frac{1}{10}$	1.0617	0.1111

$R = \frac{1}{2n}$



$R = \frac{1}{2n+1}$



Families of Curves

Definition (Families of Curves)

For A fixed, take $R = \frac{k}{n}$ and $j = n - k$. From $B(\omega)$ and $C(\omega)$, one can see that the singularities occur at $\frac{ni\pi}{j}, i = 0, 1, \dots$. The bifurcation curve i, Γ_i, with $\frac{n(i-1)\pi}{j} < \omega < \frac{ni\pi}{j}$ satisfies:

$$B_i(\omega) = \frac{A\sin(\frac{k\omega}{n}) + \omega\cos(\frac{k\omega}{n})}{\sin(\frac{j\omega}{n})}, \qquad C_i(\omega) = -\frac{A\sin(\omega) + \omega\cos(\omega)}{\sin(\frac{j\omega}{n})}$$

Now consider Γ_{i+2j} with $\mu = \omega + 2n\pi$, then

$$B_{i+2j}(\mu) = \frac{A\sin(\frac{k\mu}{n}) + \mu\cos(\frac{k\mu}{n})}{\sin(\frac{j\mu}{n})} = \frac{A\sin(\frac{k\omega}{n}) + (\omega + 2n\pi)\cos(\frac{k\omega}{n})}{\sin(\frac{j\omega}{n})}$$

$$C_{i+2j}(\mu) = -\frac{A\sin(\omega) + (\omega + 2n\pi)\cos(\omega)}{\sin(\frac{j\omega}{n})}$$

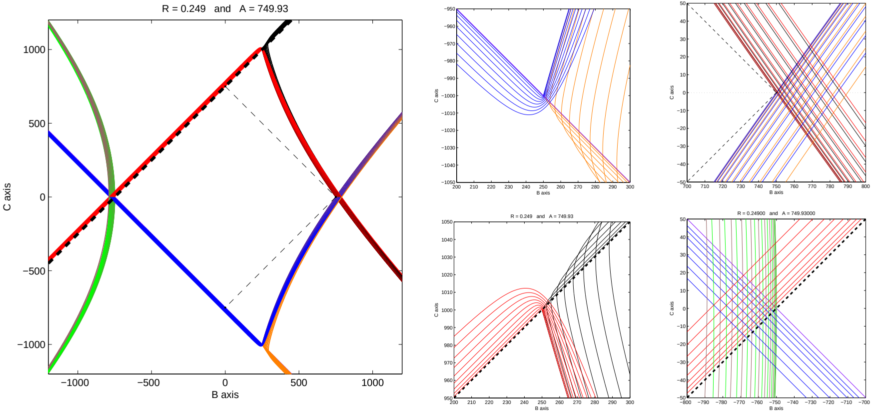
Families of Curves (cont)

Definition (Families of Curves - continued)

These equations show that $B_{i+2j}(\mu)$ follows the same trajectory as $B_i(\omega)$ with a shift of $2n\pi\cos(\frac{k\omega}{n})/\sin(\frac{j\omega}{n})$ for $\omega \in \left(\frac{(j-1)\pi}{1-R}, \frac{j\pi}{1-R}\right)$, while $C_{i+2j}(\mu)$ follows the same trajectory as $C_i(\omega)$ with a shift of $2n\pi\cos(\omega)/\sin(\frac{j\omega}{n})$ over the same values of ω. This related behavior of bifurcation curves separated by $\omega = 2n\pi$ creates $2j$ **families of curves** in the BC plane for fixed A. Thus, there is a quasi-periodicity among the bifurcation curves when R is rational.

This definition shows that $R = \frac{1}{2}$ has only 2 **families**, $R = \frac{1}{3}$ has only 4 **families**, and $R = \frac{1}{4}$ has only 6 **families**

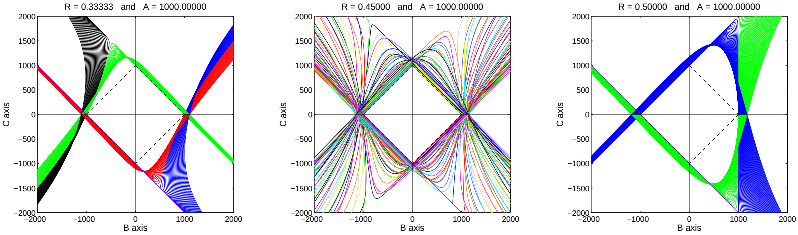
Families of Curves for $R = 0.249$ at $A_3^* = 749.93$



Ten bifurcation curves for each of the six families for $R = 0.249$ at $A_3^* = 749.93$ with close-ups at the corners of the MRS

Limited Families of Curves

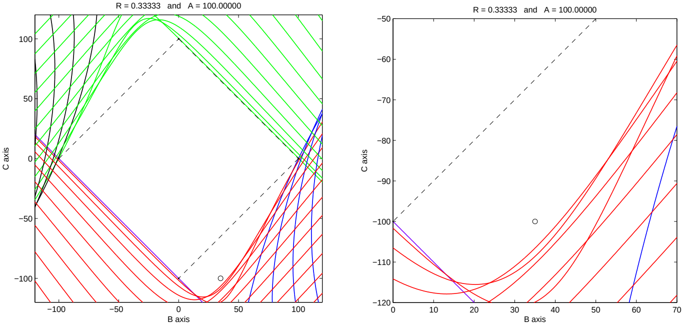
- Result of limited families is a type of resonance
- Parallel trajectories limit ability to approach the MRS
- Shows first 100 parametric curves for $A = 1000$



$R = \frac{1}{3}$ $R = 0.45$ $R = \frac{1}{2}$

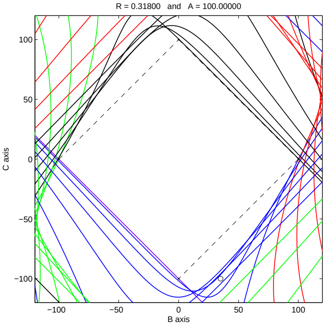
Modified Platelet Model 1

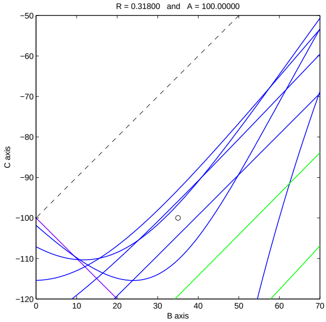
- The coefficients of the linearized model are approximately $(A, B, C) = (100, 35, -100)$
- Our bifurcation curves for $R = \frac{1}{3}$ are below



Modified Platelet Model 2

- The coefficients of the linearized model are approximately $(A, B, C) = (100, 35, -100)$
- Our bifurcation curves for $R = 0.318$ are below





Modified Platelet Model 3

Linear Analysis

- When $R = 0.318$, the model's equilibrium is outside the bifurcation curves, Γ_9, Γ_{13}, and Γ_{17}

Modified Platelet Model 3

Linear Analysis

- When $R = 0.318$, the model's equilibrium is outside the bifurcation curves, Γ_9, Γ_{13}, and Γ_{17}
- There are **3** pairs of eigenvalues with positive real part:

$$\lambda_1 = 0.1056 \pm 58.36\, i \quad \lambda_2 = 0.06238 \pm 77.43\, i \quad \lambda_3 = 0.04914 \pm 39.32\, i$$

Modified Platelet Model 3

Linear Analysis

- When $R = 0.318$, the model's equilibrium is outside the bifurcation curves, Γ_9, Γ_{13}, and Γ_{17}
- There are **3** pairs of eigenvalues with positive real part:

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Modified Platelet Model 3

Linear Analysis

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- The period is

$$\frac{2\pi}{58.36} \approx 0.108,$$

which agrees with the period in the simulation

Discussion

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- Showed an interesting application with high sensitivity to a second delay

Questions

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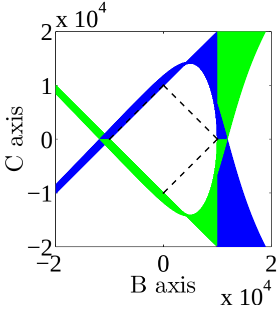
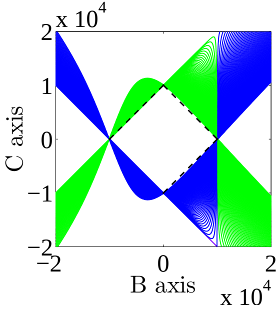


$R = \frac{1}{2}$ with 1000 curves



$R = 0.499$ with 200 curves