

Math 531 - Partial Differential Equations

Sturm-Liouville Problems

Part B

Joseph M. Mahaffy,
(jmahaffy@mail.sdsu.edu)

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

<http://jmahaffy.sdsu.edu>

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Outline

- 1 Self-Adjoint Operators
 - Lagrange's Identity
 - Green's Formula and Self-adjointness
- 2 Sturm-Liouville Properties
 - Orthogonality of Eigenfunctions
 - Real Eigenvalues
 - Unique Eigenfunctions



Regular Sturm-Liouville Problem

The **Regular Sturm-Liouville** problem satisfies:

$$\frac{d}{dx} \left(p(x) \frac{d\phi}{dx} \right) + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

with the **homogeneous BCs**:

$$\begin{aligned} \beta_1\phi(a) + \beta_2\phi'(a) &= 0, \\ \beta_3\phi(b) + \beta_4\phi'(b) &= 0, \end{aligned}$$

where (i) β_i are real, (ii) The functions $p(x)$, $q(x)$, and $\sigma(x)$ are real, continuous functions for $x \in [a, b]$ with $p(x) > 0$ and $\sigma(x) > 0$.

The following theorems will NOT be proved:

- 1 There are infinitely many **eigenvalues**.
- 2 Any **piecewise smooth** function can be expanded by the **eigenfunctions**.
- 3 Each succeeding **eigenfunction** has an additional zero.



Linear Operator

Linear Operator: Let L be the **linear differential operator**:

$$\begin{aligned} L &= \frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x), \\ L(y) &= \frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y. \end{aligned}$$

The **Sturm-Liouville differential equation** is written:

$$L(\phi) + \lambda\sigma(x)\phi = 0,$$

where λ is an **eigenvalue** and ϕ is an **eigenfunction**.

Note: The **Linear Operator** does NOT have to act on an **eigenvalue problem**.



Lagrange's Identity

Theorem (Lagrange's Identity)

Let L be the **Linear Operator**:

$$L = \frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x).$$

The following formula:

$$uL(v) - vL(u) = \frac{d}{dx} \left[p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \right],$$

is known as the **differential form of Lagrange's identity**.

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Linear Operator

Proof: This identity is readily shown

$$\begin{aligned} uL(v) - vL(u) &= u \left[\frac{d}{dx} \left(p(x) \frac{dv}{dx} \right) + q(x)v \right] - v \left[\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u \right], \\ &= u \frac{d}{dx} \left(p(x) \frac{dv}{dx} \right) - v \frac{d}{dx} \left(p(x) \frac{du}{dx} \right). \end{aligned}$$

However, from the product rule we see that

$$\begin{aligned} \frac{d}{dx} \left(u \left(p \frac{dv}{dx} \right) \right) &= u \frac{d}{dx} \left(p(x) \frac{dv}{dx} \right) + \frac{du}{dx} \cdot p \frac{dv}{dx}, \\ \frac{d}{dx} \left(v \left(p \frac{du}{dx} \right) \right) &= v \frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + \frac{dv}{dx} \cdot p \frac{du}{dx}. \end{aligned}$$

Subtracting these equations, we can insert into the expression for $uL(v) - vL(u)$ to obtain **Lagrange's identity**:

$$uL(v) - vL(u) = \frac{d}{dx} \left[p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \right]. \quad q.e.d.$$

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Green's Formula

Lagrange's identity relates to the first part of the **linear differential operator** from the Sturm-Liouville problem.

Theorem (Green's Formula)

The integration of **Lagrange's identity** give's **Green's formula**:

$$\int_a^b [uL(v) - vL(u)] dx = p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \Big|_a^b$$

for u and v continuously differentiable for $x \in (a, b)$.

This **linear differential operator** has important properties when there are **homogeneous BCs**.

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Self-adjoint

Suppose that u and v are any two functions with the additional restriction that the boundary terms vanish:

$$p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \Big|_a^b = 0.$$

Theorem (Self-adjoint)

If u and v are any two functions satisfying the same set of **homogeneous BCs** of the **regular Sturm-Liouville problem**, then

$$\int_a^b [uL(v) - vL(u)] dx = 0.$$

The **linear operator** L satisfying this condition is **self-adjoint**.

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Self-adjoint

The theorem for **self-adjointness** of the **linear operator** L extends to other **Sturm-Liouville problems**.

- 1 HW Exercises examine some **regular Sturm-Liouville problems**.
- 2 It is easy to show for **periodic BCs**.
- 3 Also, easy to show if the **Sturm-Liouville problem** is singular at $x = 0$ and has $p(0) = 0$ with $\phi(0)$ bounded.



Orthogonality of Eigenfunctions

From before

$$(\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma(x) dx = p(x) \left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx} \right) \Big|_a^b,$$

so if ϕ_m and ϕ_n satisfy the same **homogeneous BCs** the right hand side is **zero**.

Thus,

$$(\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma(x) dx = 0,$$

which says that **eigenfunctions** corresponding to different **eigenvalues** are **orthogonal** with respect to the weighting function, $\sigma(x)$.



Orthogonality of Eigenfunctions

Let λ_n and λ_m be distinct **eigenvalues** with corresponding **eigenfunctions** ϕ_n and ϕ_m , so

$$\begin{aligned} L(\phi_n) + \lambda_n \sigma(x) \phi_n &= 0, \\ L(\phi_m) + \lambda_m \sigma(x) \phi_m &= 0. \end{aligned}$$

It follows that

$$\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)] dx = (\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma(x) dx$$

By **Green's Formula**:

$$(\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma(x) dx = p(x) \left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx} \right) \Big|_a^b.$$



Real Eigenvalues

Suppose that an **eigenvalue**, λ , is **complex** and it has a corresponding **eigenfunction**, $\phi(x)$,

$$L(\phi) + \lambda \sigma \phi = 0.$$

Take the complex conjugate, so

$$\overline{L(\phi) + \lambda \sigma \phi} = \overline{L(\phi)} + \bar{\lambda} \sigma \bar{\phi} = 0.$$

However, $\overline{L(\phi)} = L(\bar{\phi})$, since the coefficients of L are real, so

$$L(\bar{\phi}) + \bar{\lambda} \sigma \bar{\phi} = 0.$$

Thus, if λ is a **complex eigenvalue** with corresponding **eigenfunction** ϕ , then $\bar{\lambda}$ is also an **eigenvalue** with **eigenfunction** $\bar{\phi}$.



Real Eigenvalues

By our **orthogonality theorem**,

$$(\lambda - \bar{\lambda}) \int_a^b \phi \bar{\phi} \sigma dx = 0.$$

However, $\phi \bar{\phi} = |\phi|^2 \geq 0$ and $\sigma > 0$.

Thus, the integral is **zero** if and only if $\phi(x) \equiv 0$, which is not an **eigenfunction**, or $\lambda - \bar{\lambda} = 0$, which implies λ is real.

Thus, **eigenvalues** of a **Sturm-Liouville problem** are **real**.



Uniqueness

Suppose that ϕ_1 and ϕ_2 are **eigenfunctions** corresponding to λ , so

$$L(\phi_1) + \lambda \phi_1 = 0 \quad \text{and} \quad L(\phi_2) + \lambda \phi_2 = 0.$$

Since λ is the same,

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = 0.$$

Lagrange's identity implies:

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx} \left[p \left(\phi_2 \frac{d\phi_1}{dx} - \phi_1 \frac{d\phi_2}{dx} \right) \right] = 0.$$

Therefore, $p \left(\phi_2 \frac{d\phi_1}{dx} - \phi_1 \frac{d\phi_2}{dx} \right)$ is a constant.



Uniqueness

Using the **Lagrange's identity**, we obtained

$$p \left(\phi_2 \frac{d\phi_1}{dx} - \phi_1 \frac{d\phi_2}{dx} \right) = C.$$

The constant $C = 0$, if we have **regular Sturm-Liouville problem** with at least one **homogeneous BC**.

It follows that

$$\phi_2 \frac{d\phi_1}{dx} - \phi_1 \frac{d\phi_2}{dx} = 0 \quad \text{or} \quad \frac{d}{dx} \left(\frac{\phi_2}{\phi_1} \right) = 0.$$

Thus, $\phi_2(x) = c\phi_1(x)$ with these BCs, so the **eigenfunction** is **unique**.



Non-Uniqueness

Earlier we showed that **periodic BCs** give both sine and cosine **eigenfunctions**, so this case does **NOT** produce **unique eigenfunctions** for a given **eigenvalue**.

Without **homogeneous BCs**, the **eigenvalue**, λ may not have a **unique eigenfunction**.

Non-uniqueness can create problems with **orthogonality** for a given **eigenvalue**.

The **Gram-Schmidt** process can be applied to create an **orthogonal set of eigenfunctions**.

Recall that $\lambda_m \neq \lambda_n$ always produces **orthogonal eigenfunctions**.

