

Math 531 - Partial Differential Equations

Vibrating String

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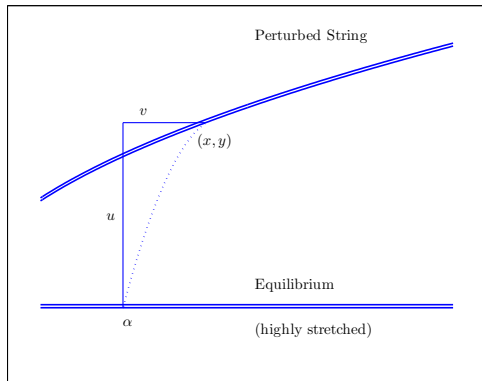
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Introduction

An important application of **PDEs** is the investigation of **vibrations** of perfectly elastic strings and membranes

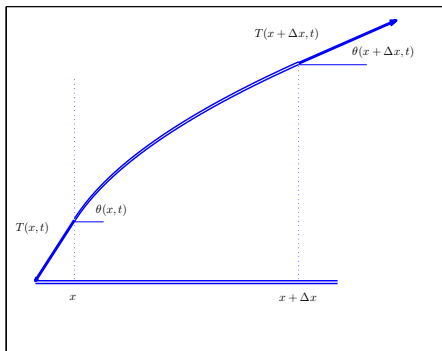


- Consider a particle at position α in a highly stretched string
- Assume a small displacement as seen above

Derivation

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Simplify by assuming the displacement is only vertical, $y = u(x, t)$



- Apply **Newton's Law** to an infinitesimally small segment of string between x and $x + \Delta x$
- Assume string has **mass density** $\rho_0(x)$, so **mass** is $\rho_0(x)\Delta x$

Derivation

Newton's Law acting on string considers all forces

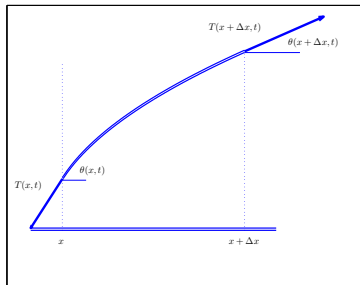
Forces include gravity, resistance, and tension - “body” forces

Assume string is *perfectly flexible*, so no bending resistance

This implies primary force is tangent to the string at all points

Tension is the tangential force with

$$\frac{dy}{dx} = \frac{\partial u}{\partial x} = \tan(\theta(x, t))$$



Derivation

Newton's Law gives $\tilde{\mathbf{F}} = m\tilde{\mathbf{a}}$, which is

$$\rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T(x + \Delta x, t) \sin(\theta(x + \Delta x, t)) \\ - T(x, t) \sin(\theta(x, t)) + \rho_0(x)\Delta x Q(\xi, t),$$

where $\xi \in [x, x + \Delta x]$ and $Q(\xi, t)$ are any “body” accelerations, such as gravity or air resistance.

Dividing by Δx and taking the limit as $\Delta x \rightarrow 0$ gives

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(T(x, t) \sin(\theta(x, t)) \right) + \rho_0(x) Q(x, t).$$

For θ “small,” let

$$\frac{\partial u}{\partial x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \sin(\theta)$$

String Equation

From previous results, obtain **String Equation**

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(T(x, t) \frac{\partial u}{\partial x} \right) + \rho_0(x) Q(x, t).$$

If the string is perfectly elastic, then $T(x, t) \approx T_0$ constant, which is equivalent to almost uniform stretching along string

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \rho_0(x) Q(x, t).$$

If the **body force** is small and **density** is constant, then

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $c^2 = \frac{T_0}{\rho_0}$.

Vibrating String - Separation of Variables

The **vibrating string** satisfies the following:

$$\text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{BC: } u(0, t) = 0, \\ u(L, t) = 0.$$

$$\text{IC: } u(x, 0) = f(x), \\ u_t(x, 0) = g(x).$$

This **vibrating string** problem or **wave equation** has fixed ends at $x = 0$ and $x = L$ and initial position, $f(x)$, and initial velocity, $g(x)$.

As before, we apply our **separation of variables** technique:

$$u(x, t) = \phi(x)h(t),$$

so

$$\phi''h = c^2\phi h'' \quad \text{or} \quad \frac{h''}{c^2h} = \frac{\phi''}{\phi} = -\lambda.$$

Vibrating String - SL Problem

The *homogeneous BCs* give:

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The **Sturm-Liouville Problem** becomes

$$\phi'' + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = 0 = \phi(L).$$

As before, we saw $\lambda \leq 0$ results in the *trivial solution*.

If we take $\lambda = \alpha^2 > 0$, then

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x),$$

where the **BCs** show $c_1 = 0$ and $\alpha = \frac{n\pi}{L}$ for nontrivial solutions.

The *eigenvalues* and *associated eigenfunctions* are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{with} \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Vibrating String - Superposition

The other *second order DE* becomes:

$$h'' + \frac{n^2\pi^2}{L^2}c^2h = 0,$$

which has the solution

$$h_n(t) = c_1 \cos\left(\frac{n\pi ct}{L}\right) + c_2 \sin\left(\frac{n\pi ct}{L}\right).$$

It follows that

$$u_n(x, t) = \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

The **Superposition principle** gives:

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

Vibrating String - ICs

The **initial position** gives:

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The **velocity** satisfies

$$u_t(x, t) = \sum_{n=1}^{\infty} \left[-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right) \right] \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

The **initial velocity** gives:

$$u_t(x, 0) = g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right),$$

where

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Physical Interpretation

Physical Interpretation: Model for vibrating string

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

- Musical instruments
- Each value of n gives a *normal mode of vibration*
- **Intensity** depends on the *amplitude*

$$A_n \cos(\omega t) + B_n \sin(\omega t) = \sqrt{A_n^2 + B_n^2} \sin(\omega t + \theta), \quad \theta = \arctan\left(\frac{A_n}{B_n}\right)$$

- Time dependence is *simple harmonic* with *circular frequency*, $\frac{n\pi c}{L}$, which is the number of oscillations in 2π units of time
- The sound produced consists of superposition of the infinite number of *natural frequencies*, $n = 1, 2, \dots$

Physical Interpretation

Physical Interpretation (cont):

- The **normal mode**, $n = 1$, is called the *first harmonic* or *fundamental mode*
- This mode has *circular frequency*, $\frac{\pi c}{L}$
- Higher natural frequencies have higher pitch
- **Fundamental frequency** varied by changing, $c = \sqrt{\frac{T_0}{\rho_0}}$
 - Tune by changing tension, T_0
 - Different ρ_0 for different strings (range of notes)
 - Musician varies pitch by varying the length L (clamping string)
- Higher harmonics for stringed instruments are all integral multiples (pleasing to the ear)

Traveling Wave

Traveling Wave: Show that the solution to the vibrating string decomposes into two waves traveling in opposite directions.

- At each t , each mode looks like a simple oscillation in x , which is a *standing wave*
- The amplitude simply varies in time
- The *standing wave* satisfies:

$$\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2} \cos\left(\frac{n\pi}{L}(x - ct)\right) - \frac{1}{2} \cos\left(\frac{n\pi}{L}(x + ct)\right)$$

- $\frac{1}{2} \cos\left(\frac{n\pi}{L}(x - ct)\right)$ produces a *traveling wave* to the right with velocity c
- $\frac{1}{2} \cos\left(\frac{n\pi}{L}(x + ct)\right)$ produces a *traveling wave* to the left with velocity $-c$
- By **superposition** (later **d'Alembert's solution**)

$$u(x, t) = R(x - ct) + S(x + ct)$$