

Math 531 - Partial Differential Equations

Separation of Variables

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Homogeneous

Heat Equation: Assume a uniform rod of length L , so that the diffusivity, specific heat, and density do not vary in x

The general **heat equation** satisfies the partial differential equation (PDE):

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{Q(x, t)}{c\rho}, \quad t > 0, \quad 0 < x < L,$$

with initial conditions (ICs):

$$u(x, 0) = f(x), \quad 0 < x < L,$$

and **Dirichlet boundary conditions** (BCs):

$$u(0, t) = T_1(t) \quad \text{and} \quad u(L, t) = T_2(t), \quad t > 0.$$

If $Q(x, t) \equiv 0$, then the PDE is **homogeneous**.

If $T_1(t) \equiv T_2(t) \equiv 0$, then the BCs are **homogeneous**.



Linearity

Definition (Linearity)

An operator \mathcal{L} is linear if and only if

$$\mathcal{L}[c_1 u_1 + c_2 u_2] = c_1 \mathcal{L}[u_1] + c_2 \mathcal{L}[u_2]$$

for any two functions u_1 and u_2 and constants c_1 and c_2 .

Define the **Heat Operator**

$$\frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} = \mathcal{L}.$$



Principle of Superposition

The following shows **linearity** of the **Heat Operator**:

$$\begin{aligned}\mathcal{L}[c_1u_1 + c_2u_2] &= \left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)(c_1u_1 + c_2u_2) \\ &= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} - kc_2\frac{\partial^2 u_2}{\partial x^2} \\ &= c_1\mathcal{L}[u_1] + c_2\mathcal{L}[u_2]\end{aligned}$$

Theorem (Principle of Superposition)

If u_1 and u_2 satisfy a **linear homogeneous equation** ($\mathcal{L}(u) = 0$), then any arbitrary linear combination, $c_1u_1 + c_2u_2$, also satisfies the **linear homogeneous equation**.

Note: Concepts of linearity and homogeneity also apply to **boundary conditions**.



Homogeneous Heat Equation

The **Heat Equation with Homogeneous Boundary Conditions**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < L,$$

with initial conditions (ICs) and **Dirichlet boundary conditions** (BCs):

$$u(x, 0) = f(x), \quad 0 < x < L, \quad \text{with} \quad u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

Separation of Variables: Developed by Daniel Bernoulli in the 1700's, we separate the temperature $u(x, t)$ into a product of a function of x and a function of t

$$u(x, t) = \phi(x)G(t)$$



Separation of Variables

Separation of Variables: With $u(x, t) = \phi(x)G(t)$, we use the **heat equation** and obtain:

$$\phi(x)\frac{dG}{dt} = k\frac{d^2\phi}{dx^2}G(t).$$

Separating the variables we have

$$\frac{1}{G}\frac{dG}{dt} = \frac{k}{\phi}\frac{d^2\phi}{dx^2} \quad \text{or} \quad \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2}$$

Since the left hand side depends only on the independent variable t and the right hand side depends only on the independent variable x , these must equal a constant

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$



Two ODEs

Thus, the **Separation of Variables** results in the **two ODEs**:

$$\frac{dG}{dt} = -\lambda kG \quad \text{and} \quad \frac{d^2\phi}{dx^2} = -\lambda\phi.$$

The **boundary conditions** with the separation assumption give:

$$u(0, t) = G(t)\phi(0) = 0 \quad \text{or} \quad \phi(0) = 0,$$

since we don't want $G(t) \equiv 0$. Also,

$$u(L, t) = G(t)\phi(L) = 0 \quad \text{or} \quad \phi(L) = 0.$$

The **Time-dependent ODE** is readily solved:

$$\begin{aligned}\frac{dG}{dt} &= -\lambda kG, \\ G(t) &= ce^{-k\lambda t}.\end{aligned}$$



Sturm-Liouville Problems

The **second ODE** is a **BVP** and is in a class we'll be calling **Sturm-Liouville problems**:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

Note: The **trivial solution** $\phi(x) \equiv 0$ always satisfies this **BVP**.

If we want to satisfy a nonzero initial condition, then we need to find **nontrivial solutions** to this BVP.

From our experience in ODEs, we can readily see there are **4 cases**:

1. $\lambda = 0$
2. $\lambda < 0$
3. $\lambda > 0$
4. λ is complex

We'll ignore Case 4 and later prove that **Sturm-Liouville problems** only have **real** λ



Sturm-Liouville Problem Cases

Consider Case 1: $\lambda = 0$, so

$$\frac{d^2\phi}{dx^2} = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The general solution to this BVP is

$$\phi(x) = c_1x + c_2.$$

We have $\phi(0) = c_2 = 0$, and $\phi(L) = c_1L = 0$ or $c_1 = 0$.

It follows that when $\lambda = 0$, the **unique solution** to the **BVP** is the **trivial solution**.



Sturm-Liouville Problem Cases

Consider Case 2: $\lambda = -\alpha^2 < 0$ with $\alpha > 0$, so

$$\frac{d^2\phi}{dx^2} - \alpha^2\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The general solution to this BVP is

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

We have $\phi(0) = c_1 = 0$, and $\phi(L) = c_2 \sinh(\alpha L) = 0$ or $c_2 = 0$, since $\sinh(\alpha L) > 0$.

It follows that when $\lambda < 0$, the **unique solution** to the **BVP** is the **trivial solution**.



Sturm-Liouville Problem Cases

Consider Case 3: $\lambda = \alpha^2 > 0$ with $\alpha > 0$, so

$$\frac{d^2\phi}{dx^2} + \alpha^2\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The general solution to this BVP is

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

We have $\phi(0) = c_1 = 0$, and $\phi(L) = c_2 \sin(\alpha L) = 0$.

It follows that either $c_2 = 0$, leading to the **trivial solution**, or $\sin(\alpha L) = 0$.

We are interested in **nontrivial solutions**, so we solve $\sin(\alpha L) = 0$, which occurs when $\alpha L = n\pi$, $n = 1, 2, \dots$ or

$$\alpha = \frac{n\pi}{L}, \quad \text{or} \quad \lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \dots$$



Eigenfunctions

We saw that if $\lambda = \alpha^2 > 0$, then the **BVP**:

$$\frac{d^2\phi}{dx^2} + \alpha^2\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0,$$

has the **nontrivial solution**,

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \dots,$$

which are called **eigenfunctions** and their associated **eigenvalues** are given by

$$\lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \dots$$

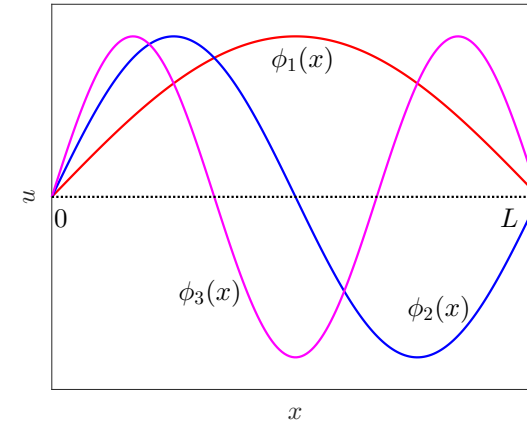
Note: $\phi_n(x)$ has $n - 1$ zeroes in $0 < x < L$, which later we'll prove is a general property



Eigenfunctions

The **Sturm-Liouville problem** from the **heat equation** with **Dirichlet BCs** generates a set of **eigenfunctions**, $\phi_n(x)$, $n = 1, 2, \dots$

Below is a graph of the first 3 **eigenfunctions**.



Product Solution

From above, the **Sturm-Liouville problem** from the **heat equation** gave the **eigenfunctions**:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \dots,$$

with associated **eigenvalues**

$$\lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \dots$$

This can be inserted into the t -equation to give:

$$G_n(t) = B_n e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

From our separation assumption, we obtain the **product solution**

$$u_n(x, t) = G_n(x, t)\phi_n(x) = B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \dots$$



Example

Example: Consider the **heat equation**:

$$\text{PDE: } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad \text{BC: } u(0, t) = 0, \\ u(10, t) = 0.$$

$$\text{IC: } u(x, 0) = 4 \sin\left(\frac{3\pi x}{10}\right),$$

From our separation of variables results, we obtain the **product solution**

$$u_n(x, t) = B_n e^{-\frac{kn^2\pi^2 t}{100}} \sin\left(\frac{n\pi x}{10}\right), \quad n = 1, 2, \dots$$

which satisfies the **BVP**

By inspection, we solve the **IC's** by taking $n = 3$ and $B_n = 4$. This gives the solution to this example as:

$$u(x, t) = 4e^{-\frac{9k\pi^2 t}{100}} \sin\left(\frac{3\pi x}{10}\right).$$



Example

Example 2: Vary the **IC** and consider the **heat equation**:

$$\text{PDE: } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad \text{BC: } u(0, t) = 0, \\ u(5, t) = 0.$$

$$\text{IC: } u(x, 0) = 3 \sin\left(\frac{3\pi x}{5}\right) + 7 \sin(\pi x),$$

With the **Principle of Superposition**, we can add our **product solutions**, $u_3(x, t) + u_5(x, t)$.

By inspection, we satisfy the **IC's** by taking $B_3 = 3$ and $B_5 = 7$. This gives the solution to this example as:

$$u(x, t) = 3e^{-\frac{9k\pi^2 t}{25}} \sin\left(\frac{3\pi x}{5}\right) + 7e^{-k\pi^2 t} \sin(\pi x).$$



Extended Superposition Principle

Extended Superposition Principle: The superposition principle can be extended to show that if u_1, u_2, \dots, u_M , are solutions of a linear homogeneous problem, then any **linear combination**

$$c_1 u_1 + c_2 u_2 + \dots + c_M u_M,$$

is also a solution.

It follows for the **homogeneous heat problem**

$$u_t = k u_{xx}, \quad u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0,$$

that we can write a solution of the form

$$u(x, t) = \sum_{n=1}^M B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$



Heat Problem with ICs

The complete **homogeneous heat problem** includes an **IC**.

Again the solution has the form:

$$u(x, t) = \sum_{n=1}^M B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right),$$

and will satisfy any **IC**, where

$$u(x, 0) = \sum_{n=1}^M B_n \sin\left(\frac{n\pi x}{L}\right) = f(x),$$

i.e., any **IC** that is a finite sum of sine functions.

What can we do about solving an arbitrary $f(x)$?



Arbitrary ICs

What if $f(x)$ is NOT a finite linear combination of appropriate sine functions?

Soon we'll learn about Fourier series

- 1 Any function with reasonable restrictions can be approximated by a linear combination of $\sin\left(\frac{n\pi x}{L}\right)$
- 2 The approximation improves with M increasing
- 3 If we consider the limit as $M \rightarrow \infty$, then with some restrictions the eigenfunctions, $\sin\left(\frac{n\pi x}{L}\right)$ in the right combination converges to $f(x)$
- 4 It remains to find the constants, B_n , such that:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$



Orthogonality of Sines

Assume $m \neq n$, integers and with some trig identities consider

$$\begin{aligned} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx &= \int_0^L \frac{\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)}{2} dx \\ &= \frac{1}{2} \left(\frac{\sin\left(\frac{(n-m)\pi x}{L}\right)}{(n-m)\pi/L} - \frac{\sin\left(\frac{(n+m)\pi x}{L}\right)}{(n+m)\pi/L} \right) \Big|_0^L \\ &= 0 \end{aligned}$$

When $m = n$, then

$$\begin{aligned} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx &= \int_0^L \frac{1 - \cos\left(\frac{2n\pi x}{L}\right)}{2} dx \\ &= \left(\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{L}\right)}{4n\pi/L} \right) \Big|_0^L \\ &= \frac{L}{2} \end{aligned}$$



Orthogonality

Definition (Orthogonality - Function Inner Product)

Whenever

$$\int_0^L A(x)B(x)dx = 0,$$

we say that the functions, $A(x)$ and $B(x)$ are **orthogonal** over the interval $[0, L]$.

Previous slide shows that the set of functions, $\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \dots$, are **orthogonal** to each other

This **orthogonal set of functions** arise from the **eigenvalue BVP**:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

Later generalize this property to any **Sturm-Liouville Problem**



Finding B_n

Consider the expression

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Use the **orthogonality of these sine functions**, so multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate $x \in [0, L]$

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \right) \sin\left(\frac{m\pi x}{L}\right) dx$$

To use orthogonality requires some analysis to allow the interchange of the integration and summation



Finding B_n

Assuming we can interchange the integration and summation,

$$\begin{aligned} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx &= \sum_{n=1}^{\infty} B_n \int_0^L \left(\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \right) dx \\ &= B_m \left(\frac{L}{2} \right), \end{aligned}$$

by the orthogonality of the sine functions

It follows that we can obtain the appropriate coefficients (Fourier) to represent an arbitrary function $f(x)$,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$



Heat Equation Example

Example: Consider the equation:

PDE: $u_t = ku_{xx}, \quad t > 0, \quad 0 < x < L,$

BC: $u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0,$

IC: $u(x, 0) = 100, \quad 0 < x < L$

From before, the solution satisfies:

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

The **Fourier coefficients** are given by

$$B_n = \frac{2}{L} \int_0^L 100 \sin\left(\frac{n\pi x}{L}\right) dx.$$



Heat Equation Example

Expanding the **Fourier coefficients**:

$$\begin{aligned} B_n &= \frac{2}{L} \int_0^L 100 \sin\left(\frac{n\pi x}{L}\right) dx = \frac{200}{L} \left(-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\right) \Big|_0^L \\ &= \frac{200}{n\pi} (1 - \cos(n\pi)) = \begin{cases} \frac{400}{n\pi}, & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases} \end{aligned}$$

Thus, the solution satisfies:

$$u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n} e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

Because of the coefficient on the exponential decay term, this solution rapidly approaches

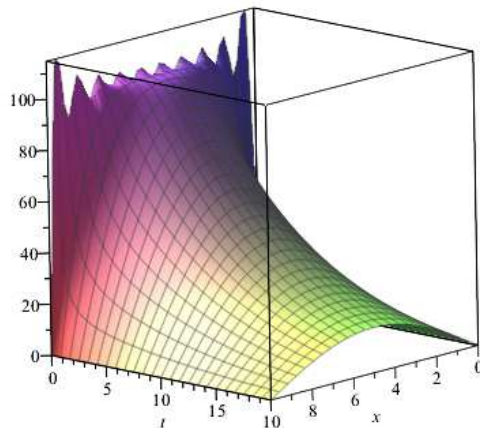
$$u(x, t) \approx \frac{400}{\pi} e^{-\frac{k\pi^2 t}{L^2}} \sin\left(\frac{\pi x}{L}\right).$$



Heat Equation with Maple

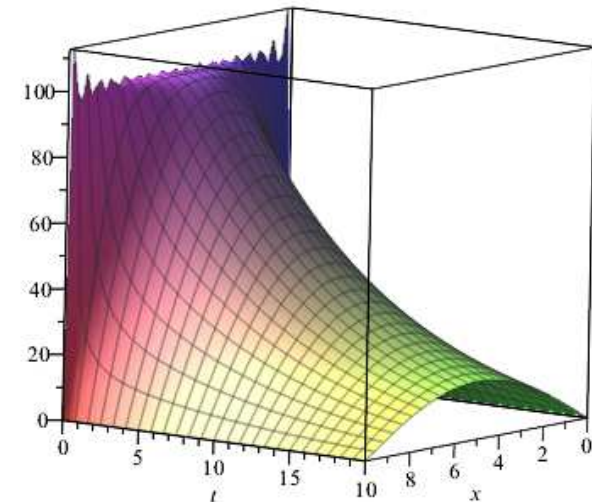
Heat Equation with Maple: Show commands and plots.

```
> u := (x,t) -> (200/Pi)*sum(((1-(-1)^n)/n)*sin(n*Pi*x/10)
    *exp(-(n*Pi/10)^2*t),n=1..20);
> plot3d(u(x,t),x=0..10,t=0..20);
```



Heat Equation with Maple

Heat Equation with Maple: Increase the sum to 60 (30 nonzero terms)



Heat Equation with MatLab

MatLab Program for Heat equation solution $u(x,t)$

```

1 % Solutions to heat flow in 1-D rod length L
2 format compact;
3 L = 10;           % length of rod
4 Temp = 100;      % Constant initial temperature
5 tfin = 20;       % final time
6 k = 1;           % heat coef of the medium
7 NptsX=151;       % number of x pts
8 NptsT=151;       % number of t pts
9 Nf=200;          % number of Fourier terms
10 x=linspace(0,L,NptsX);
11 t=linspace(0,tfin,NptsT);
12 [X,T]=meshgrid(x,t);
13 b=zeros(1,Nf);
14 U=zeros(NptsT,NptsX);
    
```

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Heat Equation with MatLab

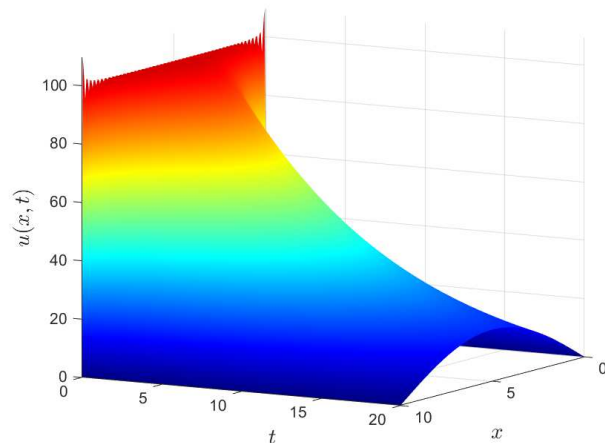
```

15 for n=1:Nf
16     b(n)=(2*Temp/(n*pi))*(1-(-1)^n); % Fourier ...
17     Un=b(n)*exp(-(n*pi*k/L)^2*T).*sin(n*pi*X/L); ...
18     % Temperature(n)
19     U=U+Un;
20 end
21 set(gca,'FontSize',[14]);
22 surf(X,T,U);
23 shading interp
24 colormap(jet)
25 xlabel('$x$', 'Fontsize',14, 'interpreter','latex');
26 ylabel('$t$', 'Fontsize',14, 'interpreter','latex');
27 zlabel('$u(x,t)$', 'Fontsize',14, 'interpreter','latex');
28 axis tight;
29 view([120 10]);
30 print -depsc heat-surf.eps
    
```

SDSU

Heat Equation with MatLab

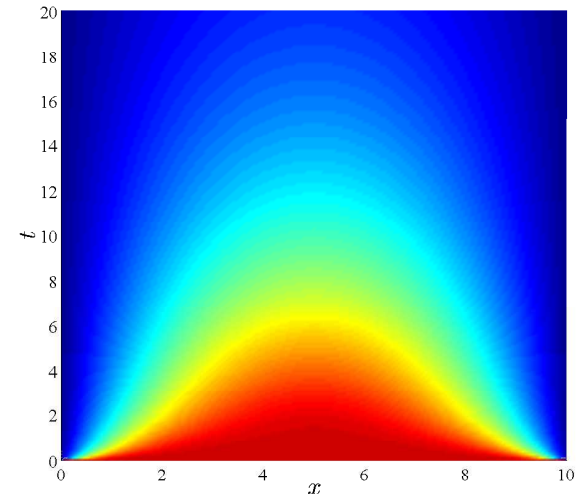
Graph of Heat Equation Solution using 200 terms with MatLab



SDSU

Heat Equation with MatLab

Changing the view to `view([0 90]);`, obtain a *heat map*



SDSU