

Math 531 - Partial Differential Equations

Fourier Transforms for PDEs

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Infinite Domain Problems

Heat Equation: Consider the PDE on an infinite domain:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty,$$

with IC

$$u(x, 0) = f(x).$$

Additionally, the IC satisfies:

$$\lim_{x \rightarrow \pm\infty} f(x) = 0.$$

The BCs are

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0.$$



Infinite Domain Problems

As before, we use *separation of variables*:

$$u(x, t) = \phi(x)g(t).$$

From the PDE, we obtain

$$\phi(x)g'(t) = k\phi''(x)g(t), \quad \text{or} \quad \frac{g'}{kg} = \frac{\phi''}{\phi} = -\lambda.$$

The *eigenvalue problem* becomes:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad |\phi(\pm\infty)| < \infty.$$

This has bounded solutions for $\lambda \geq 0$. In particular, if $\lambda = \omega^2$, then

$$\phi(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x).$$



Infinite Domain Problems

From before, any $\lambda \geq 0$ solves the *eigenvalue problem*,

so we obtain a *continuous spectrum* for $\lambda \geq 0$.

The solution to the t -dependent equation is:

$$g(t) = e^{-k\omega^2 t}.$$

Superposition principle: Since the eigenvalues form a *continuous spectrum*, the *superposition principle* requires integration over the *continuous spectrum*, rather than an infinite sum.

The solution becomes:

$$u(x, t) = \int_0^\infty [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)] e^{-k\omega^2 t} d\omega.$$



Infinite Domain Problems

It remains to show this satisfies the **IC**, so

$$u(x, 0) = f(x) = \int_0^\infty [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)] d\omega.$$

It remains to show there exist $A(\omega)$ and $B(\omega)$, which are valid for most functions, $f(x)$.

Complex exponentials: Recall that **Euler's formula** gives:

$$\cos(\omega x) = \frac{e^{i\omega x} + e^{-i\omega x}}{2} \quad \text{and} \quad \sin(\omega x) = \frac{e^{i\omega x} - e^{-i\omega x}}{2i},$$

so complex solutions are linear combinations of complex exponentials.

An alternate way to write the solution is:

$$u(x, t) = \int_{-\infty}^\infty c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega.$$

Convention uses $e^{-i\omega x}$ with $|\omega|$ being the wave number.

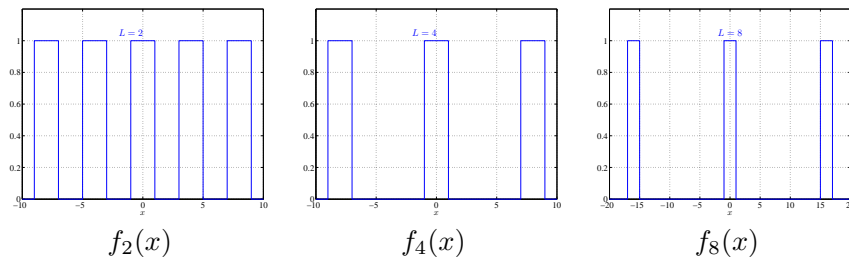


Fourier Integral

Example: Consider the periodic function:

$$f_L(x) = \begin{cases} 0, & -L < x < -1, \\ 1, & -1 < x < 1, \\ 0, & 1 < x < L, \end{cases}$$

where $f_L(x + 2Ln) = f_L(x)$ for all integers n , creating a $2L$ -periodic function.



Fourier Integral

Example: The limiting case is given by:

$$f(x) = \begin{cases} 1, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The functions above are even, so the Fourier series contains only cosine terms, so

$$f_L(x) = a_0 + \sum_{n=1}^\infty a_n \cos\left(\frac{n\pi x}{L}\right).$$

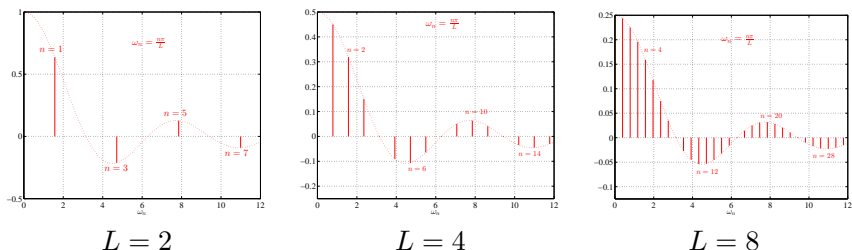
The Fourier coefficients are:

$$a_0 = \frac{1}{2L} \int_{-1}^1 1 dx = \frac{1}{L} \quad \text{and} \quad a_n = \frac{1}{L} \int_{-1}^1 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi}{L}\right).$$



Fourier Integral

Fourier coefficients: The sequence of Fourier coefficients is called the **amplitude spectrum** of $f_L(x)$ because $|a_n|$ is the maximum amplitude of $a_n \cos\left(\frac{n\pi x}{L}\right)$, where $a_n = \frac{2}{n\pi} \sin\left(\frac{n\pi}{L}\right) = \frac{2}{n\pi} \sin(\omega_n)$.



The **amplitude spectrum** becomes denser as L increases. Thus, the discrete system approaches the continuous system.



Fourier Series

Fourier Series: The Fourier series for $f_L(x)$ is

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\omega_n x) + b_n \sin(\omega_n x)), \quad \omega_n = \frac{n\pi}{L}.$$

With the Fourier coefficient formulas,

$$f_L(x) = \frac{1}{2L} \int_{-L}^L f_L(v) dv + \frac{1}{L} \sum_{n=1}^{\infty} \left[\cos(\omega_n x) \int_{-L}^L f_L(v) \cos(\omega_n v) dv + \sin(\omega_n x) \int_{-L}^L f_L(v) \sin(\omega_n v) dv \right].$$

Define $\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$, so $\frac{1}{L} = \frac{\Delta\omega}{\pi}$.



Fourier Series to Fourier Integral

With the information that the normalization $\frac{1}{L} = \frac{\Delta\omega}{\pi}$ and for all finite L , we have

$$f_L(x) = \frac{1}{2L} \int_{-L}^L f_L(v) dv + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\cos(\omega_n x) \Delta\omega \int_{-L}^L f_L(v) \cos(\omega_n v) dv + \sin(\omega_n x) \Delta\omega \int_{-L}^L f_L(v) \sin(\omega_n v) dv \right].$$

Let $L \rightarrow \infty$, then

$$f(x) = \lim_{L \rightarrow \infty} f_L(x).$$

It is plausible (assuming $f(x)$ **absolutely integrable**) that

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\cos(\omega x) \int_{-\infty}^{\infty} f(v) \cos(\omega v) dv + \sin(\omega x) \int_{-\infty}^{\infty} f(v) \sin(\omega v) dv \right] d\omega.$$

Definition (Absolutely Integrable)

A function $f(x)$ is **absolutely integrable** if the limits exists for

$$\lim_{a \rightarrow -\infty} \int_a^0 |f(x)| dx + \lim_{b \rightarrow \infty} \int_0^b |f(x)| dx.$$



Fourier Integral

In the limiting case, the **Fourier series** naturally transformed to the **Fourier Integral**:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\cos(\omega x) \int_{-\infty}^{\infty} f(v) \cos(\omega v) dv + \sin(\omega x) \int_{-\infty}^{\infty} f(v) \sin(\omega v) dv \right] d\omega,$$

$$f(x) = \int_0^{\infty} [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)] d\omega, \quad (1)$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(\omega v) dv \quad \text{and} \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(\omega v) dv.$$

Theorem (Fourier Integral)

If $f(x)$ is piecewise smooth in every finite interval and $f(x)$ is absolutely integrable, then $f(x)$ can be represented by a Fourier integral (1). At a point of discontinuity the value of the Fourier integral equals the midpoint of the left and right hand limits of $f(x)$ at that point.



Fourier Integral - Example

Example: Consider the example:

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

The **Fourier integral** representation is:

$$f(x) = \int_0^\infty [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)] d\omega,$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos(\omega v) dv = \frac{1}{\pi} \int_{-1}^1 \cos(\omega v) dv = \frac{2 \sin(\omega)}{\pi \omega}$$

and

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin(\omega v) dv = \frac{1}{\pi} \int_{-1}^1 \sin(\omega v) dv = 0.$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos(\omega x) \sin(\omega)}{\omega} d\omega = \begin{cases} 1, & |x| < 1, \\ \frac{1}{2}, & |x| = 1, \\ 0, & |x| > 1. \end{cases}$$



Fourier Transform Pair

We omitted the **complex Fourier series** earlier, but it satisfies:

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=-\infty}^\infty c_n e^{-in\pi x/L}.$$

The function $f(x)$ is $2L$ -periodic with

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{in\pi x/L} dx.$$

For periodic functions, $-L < x < L$, the allowable **wave numbers** ω are

$$\omega = \frac{n\pi}{L} = 2\pi \frac{n}{2L},$$

where the **wave lengths** are $\frac{2L}{n}$, which are integral partitions of the region length $2L$.

The distance between successive values of the **wave number** is:

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$



Fourier Transform

With the frequency $\omega = \frac{n\pi}{L}$ and the normalization $\frac{1}{2L} = \frac{\Delta\omega}{2\pi}$ follows that the **complex Fourier series** can be written:

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=-\infty}^\infty \frac{\Delta\omega}{2\pi} \left(\int_{-L}^L f(s) e^{i\omega s} ds \right) e^{-i\omega x}.$$

Fourier Transform is the limiting form as $L \rightarrow \infty$.

The values ω are the square root of the **eigenvalues**, and as $L \rightarrow \infty$, the **eigenvalues** get closer together, approaching a continuum.

Definition (Fourier Integral Identity)

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(s) e^{i\omega s} ds \right] e^{-i\omega x} d\omega.$$



Fourier Transform

Definition (Fourier Transform)

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^\infty f(s) e^{i\omega s} ds.$$

It follows that

$$\frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^\infty F(\omega) e^{-i\omega x} d\omega.$$

Note: Different authors do different things with the $\frac{1}{2\pi}$ factor, so watch the definitions carefully.

If $f(x)$ is continuous, then the **Fourier integral representation of $f(x)$** is

$$f(x) = \int_{-\infty}^\infty F(\omega) e^{-i\omega x} d\omega.$$



Fourier Transform

Definition (Fourier Transform Pair)

If $f(x)$ is continuous, then the *Fourier integral representation of $f(x)$* is

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x} d\omega,$$

where

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)e^{i\omega s} ds$$

The two equations above are called the *Fourier Transform pair*.

This relationship shows that $f(x)$ is composed of *waves* $e^{-i\omega x}$ for all wave numbers and wave lengths.

The *Fourier Transform pair* is valid provided $f(x)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Generally, we also want $f(x)$ to also be piecewise smooth, but this condition can be relaxed.



Inverse Fourier Transform

The *Gaussian function* often arises from the *diffusion operator*:

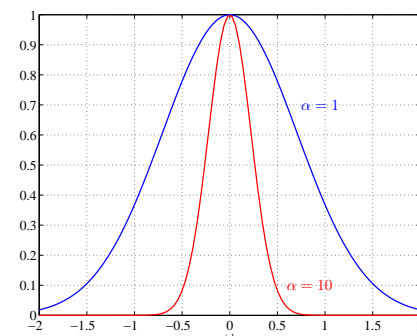
$$G(\omega) = e^{-\alpha\omega^2}.$$

The function whose *Fourier transform* is $G(\omega)$:

$$g(x) = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega$$

or

$$g(x) = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha}.$$



Fourier Transform

It follows that the inverse of a *Gaussian* is itself a *Gaussian*:

Fourier Transform Table

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x} d\omega \quad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx$$

$$f(x) = e^{-\beta x^2} \quad F(\omega) = \frac{1}{\sqrt{4\pi\beta}} e^{-\omega^2/4\beta}$$

$$f(x) = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha} \quad F(\omega) = e^{-\alpha\omega^2}$$

Derivation: Consider:

$$g(x) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega.$$

The derivative is:

$$\frac{dg}{dx} = \int_{-\infty}^{\infty} -i\omega e^{-\alpha\omega^2} e^{-i\omega x} d\omega.$$



Fourier Transform

Integration by parts with vanishing at the endpoints:

$$\frac{dg}{dx} = -\frac{i}{2\alpha} \int_{-\infty}^{\infty} \frac{d}{d\omega} (e^{-\alpha\omega^2}) e^{-i\omega x} d\omega = -\frac{x}{2\alpha} \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega = -\frac{x}{2\alpha} g(x).$$

The solution of this ODE is

$$g(x) = g(0)e^{-x^2/4\alpha}, \quad \text{where } g(0) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} d\omega.$$

Let $z = \sqrt{\alpha}\omega$ (or $dz = \sqrt{\alpha}d\omega$), so

$$g(0) = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\frac{\pi}{\alpha}}.$$

Note: There are two ways to solve this (one involves complex variables):

$$\left(\int_{-\infty}^{\infty} e^{-z^2} dz \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr d\theta = \pi.$$

