

Math 531 - Partial Differential Equations

PDEs - Higher Dimensions

Cylinder

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More on Bessel Functions

Bessel's Equation can be written:

$$\frac{d^2\phi}{dz^2} = -\left(1 - \frac{m^2}{z^2}\right)\phi - \frac{1}{z}\frac{d\phi}{dz},$$

which can be compared to the **damped-spring-mass** system:

$$\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}.$$

- 1 Bessel's equation behaves like a time-varying frictional force ($c \sim 1/t$) that gets weaker with time (less than exponential decay).
- 2 Bessel's equation behaves like a restoring force ($k \sim (1 - m^2/z^2)$) approaches constant oscillation.



Outline

- 1 More on Bessel Functions
- 2 Laplace's Equation - Cylinder
 - Problem 1 & 2 - Bottom & Top Nonhomogeneous
 - Problem 3 - Side Nonhomogeneous
 - Modified Bessel Functions
- 3 Spherical Problems and Legendre Polynomials
 - Associated Legendre Polynomials
 - Legendre Polynomials
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- 4 Laplace in Spherical Cavity



More on Bessel Functions

Asymptotic Behavior of Bessel's Equation

Small z

$$\begin{aligned} J_0(z) &\approx 1 & Y_0(z) &\approx \frac{2}{\pi} \ln(z) \\ J_1(z) &\approx \frac{1}{2}z & Y_1(z) &\approx -\frac{2}{\pi}z^{-1} \\ J_2(z) &\approx \frac{1}{8}z^2 & Y_2(z) &\approx -\frac{4}{\pi}z^{-2} \end{aligned}$$

Large z , as $z \rightarrow \infty$

$$\begin{aligned} J_m(z) &\approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) \\ Y_m(z) &\approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) \end{aligned}$$

The **zeroes** are asymptotically separated by π .



Vibrating Circular Membrane

Laplace's Equation - Cylinder: The PDE satisfies:

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

BC: Bottom

$$u(r, \theta, 0) = \alpha(r, \theta),$$

BC: Top

$$u(r, \theta, H) = \beta(r, \theta),$$

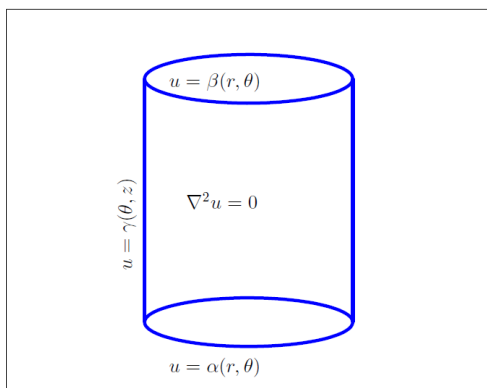
BC: Side

$$u(a, \theta, z) = \gamma(\theta, z).$$

BC: Implicit (Homogeneous)

Periodic in θ and
 Bounded $r \rightarrow 0$.

Break the problem into
3 problems each with
2 homogeneous conditions.



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Laplace's Equation - Cylinder

Problem 1: Let the **Top** and **Side** be **homogeneous** with only the **nonhomogeneous** condition:

$$u_1(r, \theta, 0) = \alpha(r, \theta).$$

The boundedness as $r \rightarrow 0$ and periodicity in the θ direction provides the other homogeneous conditions.

Use **Separation of Variables** in **Laplace's Equation** with:

$$u_1(r, \theta, z) = \phi(r)g(\theta)h(z),$$

so

$$\frac{gh}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{\phi h}{r^2} \frac{d^2 g}{d\theta^2} + \phi g \frac{d^2 h}{dz^2} = 0.$$

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Laplace's Equation - Cylinder

Separation of Variables gives

$$\frac{1}{r\phi} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{1}{r^2 g} \frac{d^2 g}{d\theta^2} = -\frac{h''}{h} = -\lambda,$$

which gives the z -equation:

$$h'' - \lambda h = 0.$$

Multiply by r^2 and rearrange to obtain:

$$\frac{r}{\phi} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \lambda r^2 = -\frac{g''}{g} = \mu, \quad \text{or} \quad g'' + \mu g = 0.$$

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Laplace's Equation - Cylinder

1st Sturm-Liouville Problem is:

$$g'' + \mu g = 0, \quad \text{with} \quad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

As seen before, this problem has **eigenvalues**, $\mu_m = m^2$,
 $m = 0, 1, 2, \dots$ and corresponding **eigenfunctions**:

$$g_0(\theta) = a_0 \quad \text{and} \quad g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$

2nd Sturm-Liouville Problem is:

$$\frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \left(\lambda r - \frac{m^2}{r} \right) \phi = 0, \quad \text{with} \quad \phi(a) = 0 \quad \text{and} \quad |\phi(0)| < \infty,$$

which is **Bessel's equation of order m** .

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Laplace's Equation - Cylinder

The **2nd Sturm-Liouville Problem** in r has the general solution:

$$\phi(r) = c_1 J_m(\sqrt{\lambda}r) + c_2 Y_m(\sqrt{\lambda}r).$$

Since $|\phi(0)| < \infty$, we have $c_2 = 0$. The other **homogeneous BC** gives:

$$\phi(a) = c_1 J_m(\sqrt{\lambda_{mn}a}) = 0.$$

As seen before, this has **eigenvalues** and **eigenfunctions**:

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2, \quad \phi_{mn}(r) = J_m(z_{mn}r/a), \quad m = 0, 1, 2, \dots \quad n = 1, 2, \dots,$$

where z_{mn} is the n^{th} zero satisfying $J_m(z_{mn}) = 0$.



Laplace's Equation - Cylinder

With $\lambda_{mn} > 0$, we solve

$$h'' - \lambda h = 0,$$

to obtain

$$h(z) = d_1 \cosh(\sqrt{\lambda_{mn}}(H - z)) + d_2 \sinh(\sqrt{\lambda_{mn}}(H - z)).$$

However, $h(H) = 0$, so $d_1 = 0$ or $h(z) = \sinh(\sqrt{\lambda_{mn}}(H - z))$.

We apply the **superposition principle** to obtain u_1 :

$$u_1(r, \theta, z) = \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}}r) \sinh(\sqrt{\lambda_{0n}}(H - z)) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta) \right) \cdot J_m(\sqrt{\lambda_{mn}}r) \sinh(\sqrt{\lambda_{mn}}(H - z)).$$



Laplace's Equation - Cylinder

Fourier coefficients are found with the **nonhomogeneous BC**:

$$u_1(r, \theta, 0) = \alpha(r, \theta) = \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}}r) \sinh(\sqrt{\lambda_{0n}}H) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta) \right) \cdot J_m(\sqrt{\lambda_{mn}}r) \sinh(\sqrt{\lambda_{mn}}H).$$

With **orthogonality**, we find

$$A_{0n} = \frac{\int_{-\pi}^{\pi} \int_0^a \alpha(r, \theta) J_0(\sqrt{\lambda_{0n}}r) r dr d\theta}{2\pi \sinh(\sqrt{\lambda_{0n}}H) \int_0^a J_0^2(\sqrt{\lambda_{0n}}r) r dr},$$

and

$$A_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a \alpha(r, \theta) \cos(m\theta) J_m(\sqrt{\lambda_{mn}}r) r dr d\theta}{\pi \sinh(\sqrt{\lambda_{mn}}H) \int_0^a J_m^2(\sqrt{\lambda_{mn}}r) r dr},$$



Laplace's Equation - Cylinder

and

$$B_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a \alpha(r, \theta) \sin(m\theta) J_m(\sqrt{\lambda_{mn}}r) r dr d\theta}{\pi \sinh(\sqrt{\lambda_{mn}}H) \int_0^a J_m^2(\sqrt{\lambda_{mn}}r) r dr}.$$

It is easy to see that almost identical computations hold for u_2 where the **nonhomogeneous BC** is the top, $u_2(r, \theta, H) = \beta(r, \theta)$.

The **2 Sturm-Liouville problems** are identical to the ones for u_1 , so the only difference is solving the z -dependent equation:

$$h'' - \lambda_{mn}h = 0, \quad \text{with } h(0) = 0.$$

This has the solution:

$$h(z) = c_1 \sinh(\sqrt{\lambda_{mn}}z).$$



Laplace's Equation - Cylinder

It follows that

$$u_2(r, \theta, z) = \sum_{n=1}^{\infty} C_{0n} J_0(\sqrt{\lambda_{0n}} r) \sinh(\sqrt{\lambda_{0n}} z) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(C_{mn} \cos(m\theta) + D_{mn} \sin(m\theta) \right) J_m(\sqrt{\lambda_{mn}} r) \sinh(\sqrt{\lambda_{mn}} z).$$

The **Fourier coefficients** from the condition $\beta(r, \theta)$ are:

$$C_{0n} = \frac{\int_{-\pi}^{\pi} \int_0^a \beta(r, \theta) J_0(\sqrt{\lambda_{0n}} r) r dr d\theta}{2\pi \sinh(\sqrt{\lambda_{0n}} H) \int_0^a J_0^2(\sqrt{\lambda_{0n}} r) r dr},$$

and

$$C_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a \beta(r, \theta) \cos(m\theta) J_m(\sqrt{\lambda_{mn}} r) r dr d\theta}{\pi \sinh(\sqrt{\lambda_{mn}} H) \int_0^a J_m^2(\sqrt{\lambda_{mn}} r) r dr},$$

and

$$D_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a \beta(r, \theta) \sin(m\theta) J_m(\sqrt{\lambda_{mn}} r) r dr d\theta}{\pi \sinh(\sqrt{\lambda_{mn}} H) \int_0^a J_m^2(\sqrt{\lambda_{mn}} r) r dr}.$$

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Laplace's Equation - Cylinder

The **cylinder problem** for u_3 , where the **nonhomogeneous BC** is the side, $u_3(a, \theta, z) = \gamma(\theta, z)$, must be handled differently.

With the side nonhomogeneous, the r -dependent equation can no longer be one of the **2 Sturm-Liouville problems**.

The **separation of variables** for $u_3(r, \theta, z) = \phi(r)g(\theta)h(z)$ gives:

$$\frac{1}{r\phi} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{1}{r^2 g} \frac{d^2 g}{d\theta^2} = -\frac{h''}{h} = \lambda.$$

Now the **1st Sturm-Liouville problem** is:

$$h'' + \lambda h = 0, \quad \text{with } h(0) = 0 \quad \text{and} \quad h(H) = 0.$$

From before, this has the **eigenvalues** and **eigenfunctions**:

$$\lambda_n = \frac{n^2 \pi^2}{H^2} \quad \text{with} \quad h_n(z) = \sin\left(\frac{n\pi z}{H}\right).$$

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Laplace's Equation - Cylinder

Multiplying by r^2 and rearranging the **separation equation** gives:

$$\frac{r}{\phi} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) - \lambda_n r^2 = -\frac{g''}{g} = \mu, \quad \text{or} \quad g'' + \mu g = 0.$$

The **2nd Sturm-Liouville Problem** is now:

$$g'' + \mu g = 0, \quad \text{with } g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi),$$

which as before has **eigenvalues**, $\mu_m = m^2$, $m = 0, 1, 2, \dots$ and corresponding **eigenfunctions**:

$$g_0(\theta) = a_0 \quad \text{and} \quad g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$

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Laplace's Equation - Cylinder

Returning to the **separation equation**, we obtain the **3rd ODE**, which is given by:

$$\frac{d}{dr} \left(r \frac{d\phi}{dr} \right) - \left(\frac{n^2 \pi^2}{H^2} r + \frac{m^2}{r} \right) \phi = 0, \quad \text{with } |\phi(0)| < \infty,$$

which because of the sign is **not Bessel's equation**.

Let $z = \frac{n\pi}{H} r$, then the **3rd ODE** can be written:

$$z^2 \frac{d^2 \phi}{dz^2} + z \frac{d\phi}{dz} - (z^2 + m^2) \phi = 0,$$

which is known as **modified Bessel's equation**.

This has the solution:

$$\phi(r) = c_1 K_m\left(\frac{n\pi}{H} r\right) + c_2 I_m\left(\frac{n\pi}{H} r\right).$$

The condition that $|\phi(0)| < \infty$ implies that $c_1 = 0$, as $K_m(z) \rightarrow \infty$ as $z \rightarrow 0$. ($I_m(z)$ behaves as z^m as $z \rightarrow 0$.)

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Laplace's Equation - Cylinder

The **superposition principle** gives

$$u_3(r, \theta, z) = \sum_{n=1}^{\infty} E_{0n} I_0\left(\frac{n\pi}{H}r\right) \sin\left(\frac{n\pi}{H}z\right) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(E_{mn} \cos(m\theta) + F_{mn} \sin(m\theta)\right) I_m\left(\frac{n\pi}{H}r\right) \sin\left(\frac{n\pi}{H}z\right).$$

The **Fourier coefficients** from the condition $\gamma(\theta, z)$ are:

$$E_{0n} = \frac{\int_{-\pi}^{\pi} \int_0^H \gamma(\theta, z) \sin\left(\frac{n\pi}{H}z\right) dz d\theta}{\pi H I_0\left(\frac{n\pi}{H}a\right)},$$

and

$$E_{mn} = \frac{2 \int_{-\pi}^{\pi} \int_0^H \gamma(\theta, z) \cos(m\theta) \sin\left(\frac{n\pi}{H}z\right) dz d\theta}{\pi H I_m\left(\frac{n\pi}{H}a\right)},$$

and

$$F_{mn} = \frac{2 \int_{-\pi}^{\pi} \int_0^H \gamma(\theta, z) \sin(m\theta) \sin\left(\frac{n\pi}{H}z\right) dz d\theta}{\pi H I_m\left(\frac{n\pi}{H}a\right)}.$$

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Modified Bessel Functions

However, $K_m(z)$ is **singular** as $z \rightarrow 0$, and it can be shown that

$$K_m(z) \sim \begin{cases} \ln(z), & m = 0, \\ \frac{1}{2}(m-1)! \left(\frac{1}{2}z\right)^{-m}, & m \neq 0. \end{cases}$$

So significantly, $K_m(z)$ **decays exponentially** as $z \rightarrow \infty$, but is **singular** as $z \rightarrow 0$.

The **Modified Bessel Function** is uniquely defined such that

$$I_m(z) \sim \frac{1}{m!} \left(\frac{1}{2}z\right)^m,$$

as $z \rightarrow 0$.

However, as $z \rightarrow \infty$, it is a linear combination of the independent solutions, which behave like

$$I_m(z) \sim \sqrt{\frac{1}{2\pi z}} e^z.$$

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Modified Bessel Functions

Modified Bessel's functions satisfy:

$$z^2 \frac{d^2 \phi}{dz^2} + z \frac{d\phi}{dz} - (z^2 + m^2)\phi = 0,$$

We could write this equation:

$$\frac{d^2 \phi}{dz^2} = -\frac{1}{z} \frac{d\phi}{dz} + \left(1 + \frac{m^2}{z^2}\right) \phi,$$

which for large z gives:

$$\frac{d^2 \phi}{dz^2} \approx \phi.$$

This **differential equation** has solutions, like e^x and e^{-x} .

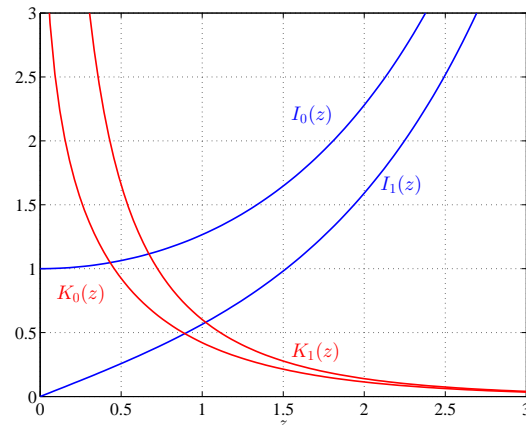
In fact, it can be shown that only one **linearly independent solution** decays as $z \rightarrow \infty$, and we define this solution:

$$K_m(z) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{z^{1/2}}.$$

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Modified Bessel Functions

So significantly, $I_m(z)$ **grows exponentially** as $z \rightarrow \infty$, but is well-behaved at $z = 0$. Below is the graph of some of the **modified Bessel functions**.



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Spherical Problems

The **Heat** or **Wave** equations:

$$\frac{\partial u}{\partial t} = k \nabla^2 u \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

can use the **separation of variables** $u(\rho, \theta, \phi, t) = w(\rho, \theta, \phi)h(t)$ to obtain either

$$\frac{h'}{kh} = \frac{\nabla^2 w}{w} = -\lambda \quad \text{or} \quad \frac{h''}{c^2 h} = \frac{\nabla^2 w}{w} = -\lambda.$$

Thus, we have the **time-equation**:

$$h' + \lambda kh = 0 \quad \text{or} \quad h'' + \lambda c^2 h = 0.$$

The **space-equation** is:

$$\nabla^2 w + \lambda w = 0.$$



Spherical Problems

After removing the ρ -equation, the θ and ϕ parts are separated to give:

$$-\frac{\sin \phi}{g} \frac{d}{d\phi} \left(\sin \phi \frac{dg}{d\phi} \right) - \mu \sin^2 \phi = \frac{q''}{q} = -\gamma.$$

The **1st Sturm-Liouville problem** in θ is:

$$q'' + \gamma q = 0, \quad \text{with BCs } q(-\pi) = q(\pi) \quad \text{and} \quad q'(-\pi) = q'(\pi),$$

which has **eigenvalues** and **eigenfunctions**

$$\gamma_0 = 0 \quad \text{and} \quad q_0(\theta) = a_0,$$

and

$$\gamma_m = m^2 \quad \text{and} \quad q_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$



Spherical Problems

In spherical coordinates the spatial problem is

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial w}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial w}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 w}{\partial \theta^2} + \lambda w = 0.$$

Once again we **separate variables** with $w(\rho, \theta, \phi) = f(\rho)q(\theta)g(\phi)$ and multiply $\rho^2/(fqq)$, then the spatial equation becomes:

$$\frac{1}{f} \frac{d}{d\rho} \left(\rho^2 \frac{df}{d\rho} \right) + \lambda \rho^2 = -\frac{1}{g \sin \phi} \frac{d}{d\phi} \left(\sin \phi \frac{dg}{d\phi} \right) - \frac{1}{q \sin^2 \phi} \frac{d^2 q}{d\theta^2} = \mu.$$

The ρ -equation is

$$\frac{d}{d\rho} \left(\rho^2 \frac{df}{d\rho} \right) + (\lambda \rho^2 - \mu) f = 0,$$

which is almost **Bessel's equation**.



Associated Legendre Polynomials

The **2nd Sturm-Liouville problem** in ϕ is:

$$\frac{d}{d\phi} \left(\sin \phi \frac{dg}{d\phi} \right) + \left(\mu \sin \phi - \frac{m^2}{\sin \phi} \right) g = 0, \quad 0 \leq \phi \leq \pi,$$

with the **singular BCs** $g(0)$ and $g(\pi)$ **bounded**.

This **SL**-problem is related to **associated Legendre polynomials**.

We make the change of variables $x = \cos(\phi)$, $-1 \leq x \leq 1$, so

$$\frac{d}{d\phi} = \frac{dx}{d\phi} \frac{d}{dx} = -\sin(\phi) \frac{d}{dx}.$$

In the **associated Legendre equation** with the change of variables, the first term is

$$-\sin \phi \frac{d}{dx} \left(-\sin^2 \phi \frac{dg}{dx} \right) = \sin \phi \frac{d}{dx} \left((1 - \cos^2 \phi) \frac{dg}{dx} \right) = \sin \phi \frac{d}{dx} \left((1 - x^2) \frac{dg}{dx} \right).$$



Associated Legendre Polynomials

We divide the **associated Legendre equation** by $\sin(\phi)$ and obtain

$$\frac{d}{dx} \left((1-x^2) \frac{dg}{dx} \right) + \left(\mu - \frac{m^2}{\sin^2 \phi} \right) g = 0,$$

which becomes

$$\frac{d}{dx} \left((1-x^2) \frac{dg}{dx} \right) + \left(\mu - \frac{m^2}{(1-x^2)} \right) g = 0.$$

This is a **Sturm-Liouville problem with regular singular points** at $x = \pm 1$ (or $\phi = 0, \pi$) the **poles**.

By writing the equation

$$g'' - \frac{2x}{(x+1)(x-1)} g' + \left(\frac{\mu(x^2-1) - m^2}{(x+1)^2(x-1)^2} \right) g = 0,$$

it is easy to see that $x = 1$ and -1 are **regular singular points**.



Associated Legendre Polynomials

The **associated Legendre equation** is often written:

$$\frac{d}{dx} \left((1-x^2) \frac{dg}{dx} \right) + \left(n(n+1) - \frac{m^2}{(1-x^2)} \right) g = 0,$$

and its **linearly independent solutions (associated Legendre functions)** are written:

$$g(x) = c_1 P_n^m(x) + c_2 Q_n^m(x).$$

It can be shown that when n is not an integer, then both solutions are unbounded at either $x = 1$ or $x = -1$.

When n is an integer, then $P_n^m(x)$ is a polynomial, while $Q_n^m(x)$ is unbounded at both $x = 1$ and $x = -1$.

Thus, we concentrate our studies on the **associated Legendre polynomials**, $P_n^m(x)$, for our physical problem.



Legendre Polynomials

If $m = 0$ (no θ dependence), **cylindrically symmetric, Legendre equation** is given by:

$$\frac{d}{dx} \left((1-x^2) \frac{dg}{dx} \right) + n(n+1)g = 0.$$

Let $g(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$\frac{d}{dx} \left((1-x^2) \sum_{k=1}^{\infty} a_k k x^{k-1} \right) + n(n+1) \sum_{k=0}^{\infty} a_k x^k = 0.$$

or

$$\sum_{k=2}^{\infty} a_k k(k-1) x^{k-2} - \sum_{k=0}^{\infty} a_k k(k+1) x^k + n(n+1) \sum_{k=0}^{\infty} a_k x^k = 0.$$

or

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k - \sum_{k=0}^{\infty} a_k (k(k+1) - n(n+1)) x^k = 0.$$



Legendre Polynomials

The power series given by

$$\sum_{k=0}^{\infty} \left(a_{k+2} (k+2)(k+1) - a_k (k(k+1) - n(n+1)) \right) x^k = 0,$$

has the **recurrence relation**:

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k = -\frac{(n-k)(1+n+k)}{(k+2)(k+1)} a_k,$$

where a_0 and a_1 are arbitrary.

It is easy to see by the **ratio test** that the series above converges for $|x| < 1$.

When $|x| = \pm 1$, this series diverges unless n is an integer, then one solution of the power series is a **polynomial**, so **converges**.



Legendre Polynomials

It follows that we can write

$$g = a_0 \left(1 - \frac{n(n+1)}{2 \cdot 1} x^2 + \frac{(n-2)(n+3)(n+1)n}{4!} x^4 - \dots \right) + a_1 \left(x - \frac{(n-1)(n+2)}{3 \cdot 2} x^3 + \frac{(n-3)(n+4)(n-1)(n+2)}{4!} x^5 - \dots \right).$$

The first **6 Legendre polynomials** are:

$$\begin{aligned} n = 0 & \quad P_0(x) = 1, \\ n = 1 & \quad P_1(x) = x, \\ n = 2 & \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \\ n = 3 & \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \\ n = 4 & \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \\ n = 5 & \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned}$$

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Legendre Polynomials

One method of generating *Legendre polynomials* is *Rodriguez formula*:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Since $x = \cos(\phi)$, the first three *Legendre polynomials* in ϕ are:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x = \cos(\phi), \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) = \frac{1}{4}(3 \cos(2\phi) + 1). \end{aligned}$$

The *orthogonality* has a *weighting function* $\sigma(x) = 1$ ($\sigma(\phi) = \sin(\phi)$) and satisfies:

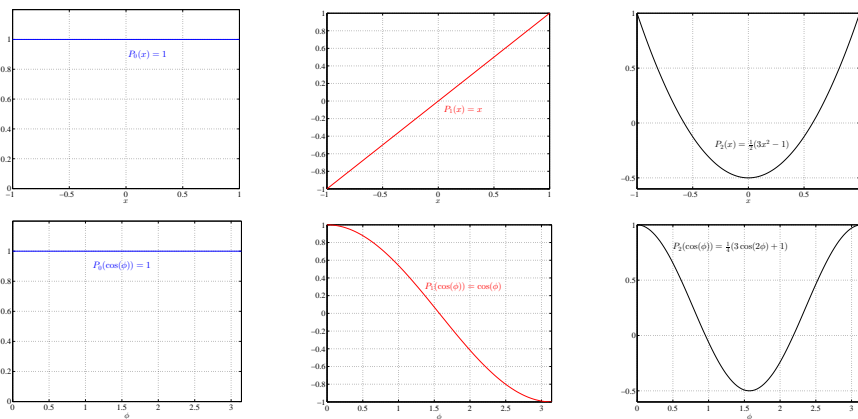
$$\int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} 0, & n \neq m, \\ \frac{2}{2n+1}, & n = m. \end{cases}$$

Uses the recurrence relation and integration by parts.

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Legendre Polynomials

Graphs of the first **3 Legendre polynomials**.



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Associated Legendre Polynomials

If $m > 0$, then the *associated Legendre polynomials* can be found with the formula:

$$g(x) = P_n^m(x) = (x^2 - 1)^{m/2} \frac{d^m}{dx^m} P_n(x),$$

where $n \geq m$ to avoid $g(x) = 0$ and $P_n(x)$ is the *Legendre polynomial of order n*.

With these formulas, we have solved for $q(\theta)$ and $g(\phi)$ for the *spherical problem*.

Remains to solve the *radial* part of this problem.

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Radial Eigenvalue Problem

If the original *spherical problem* has *homogeneous BCs*, $u(a, \theta, \phi, t) = 0$, then the 3rd *Sturm-Liouville problem* is

$$\frac{d}{d\rho} \left(\rho^2 \frac{df}{d\rho} \right) + \left(\lambda \rho^2 - n(n+1) \right) f = 0, \quad f(a) = 0,$$

which is restricted to $n \geq m$ for fixed m .

This is almost *Bessel's equation*, and it has the solution *Spherical Bessel's function*:

$$f(\rho) = \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda} \rho),$$

which are bounded at $\rho = 0$.

The *eigenvalues* satisfy $J_{n+1/2}(\sqrt{\lambda} a) = 0$, so the k^{th} zero is

$$z_{k,n+1/2} = \sqrt{\lambda_{k,n}} a.$$

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Radial Eigenvalue Problem

Spherical Bessel functions satisfy

$$x^{-1/2} J_{n+1/2}(x) = x^n \left(-\frac{1}{x} \frac{d}{dx} \right)^n \left(\frac{\sin(x)}{x} \right).$$

The *superposition principle* gives:

$$\begin{aligned} u(\rho, \theta, \phi, t) &= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} f(\rho) q(\theta) g(\phi) h(t) \\ &= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left\{ \begin{array}{l} \cos(c\sqrt{\lambda_{k,n}} t) \\ \sin(c\sqrt{\lambda_{k,n}} t) \end{array} \right\} \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\sqrt{\lambda_{k,n}} \rho) \left\{ \begin{array}{l} 1 \\ \cos(m\theta) \\ \sin(m\theta) \end{array} \right\} P_n^m(\cos(\phi)). \end{aligned}$$

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Laplace in Spherical Cavity

Consider *Laplace's equation in a spherical cavity*:

$$\nabla^2 u = 0, \quad \text{with } u(a, \theta, \phi) = F(\theta, \phi).$$

In spherical coordinates the spatial problem is

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Once again we *separate variables* with $u(\rho, \theta, \phi) = f(\rho)q(\theta)g(\phi)$ and multiply $\rho^2/(fgg)$, then the spatial equation becomes:

$$\frac{1}{f} \frac{d}{d\rho} \left(\rho^2 \frac{df}{d\rho} \right) = -\frac{1}{g \sin \phi} \frac{d}{d\phi} \left(\sin \phi \frac{dg}{d\phi} \right) - \frac{1}{q \sin^2 \phi} \frac{d^2 q}{d\theta^2} = \nu.$$

The ρ -equation is

$$\frac{d}{d\rho} \left(\rho^2 \frac{df}{d\rho} \right) - \nu f = 0.$$

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Laplace in Spherical Cavity

The *Sturm-Liouville problems* are in θ and ϕ .

The θ and ϕ parts are separated to give:

$$-\frac{\sin \phi}{g} \frac{d}{d\phi} \left(\sin \phi \frac{dg}{d\phi} \right) - \nu \sin^2 \phi = \frac{q''}{q} = -\mu.$$

The 1st *Sturm-Liouville problem* in θ is:

$$q'' + \mu q = 0, \quad \text{with BCs } q(-\pi) = q(\pi) \quad \text{and} \quad q'(-\pi) = q'(\pi),$$

which has *eigenvalues* and *eigenfunctions*

$$\mu_0 = 0 \quad \text{and} \quad q_0(\theta) = a_0,$$

and

$$\mu_m = m^2 \quad \text{and} \quad q_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$

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Laplace in Spherical Cavity

The 2^{nd} **Sturm-Liouville problem** in ϕ is:

$$\frac{d}{d\phi} \left(\sin \phi \frac{dg}{d\phi} \right) + \left(\nu \sin \phi - \frac{m^2}{\sin \phi} \right) g = 0, \quad 0 \leq \phi \leq \pi,$$

with the **singular BCs** $g(0)$ and $g(\pi)$ **bounded**.

As seen before, this **SL**-problem is related to **associated Legendre polynomials**.

The solution to this **eigenvalue problem** is **eigenvalues**, $\nu = n(n+1)$ and associated **eigenfunctions**:

$$g(\phi) = P_n^m(\cos(\phi)).$$

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Laplace in Spherical Cavity

The **radial equation** satisfies:

$$\frac{d}{d\rho} \left(\rho^2 \frac{df}{d\rho} \right) - n(n+1)f = 0.$$

This is an **equidimensional** or **Euler problem**, so attempt solutions of the form:

$$f(\rho) = \rho^r.$$

The result is:

$$\frac{d}{d\rho} (\rho^2 r \rho^{r-1}) - n(n+1)\rho^r = 0.$$

This gives

$$\rho^r (r(r+1) - n(n+1)) = (r^2 + r - n(n+1))\rho^r = 0.$$

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Laplace in Spherical Cavity

The above equation is factored to give

$$r^2 + r - n(n+1) = (r-n)(r+n+1) = 0, \quad \text{or} \quad r = n, -(n+1).$$

It follows that

$$f(\rho) = c_1 \rho^n + c_2 \rho^{-(n+1)}.$$

Since the solution is **bounded** at $\rho = 0$, it follows that $c_2 = 0$.

The **superposition principle** gives:

$$u(\rho, \theta, \phi) = \sum_{n=0}^{\infty} A_{0n} \rho^n P_n(\cos(\phi)) + \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \rho^n (A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)) P_n^m(\cos(\phi)).$$

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Laplace in Spherical Cavity

The **BC** at $\rho = a$ gives:

$$F(\theta, \phi) = \sum_{n=0}^{\infty} A_{0n} a^n P_n(\cos(\phi)) + \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} a^n (A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)) P_n^m(\cos(\phi)).$$

Recall that the **Sturm-Liouville problem** in ϕ was

$$\frac{d}{d\phi} \left(\sin(\phi) \frac{dg}{d\phi} \right) + \left(n(n+1) \sin(\phi) - \frac{m^2}{\sin(\phi)} \right) g = 0,$$

so the weighting function is $\sigma(\phi) = \sin(\phi)$.

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Laplace in Spherical Cavity

The *Fourier coefficients* are readily found using *orthogonality*, so

$$A_{0n} = \frac{\int_{-\pi}^{\pi} \int_0^{\pi} F(\theta, \phi) P_n(\cos(\phi)) \sin(\phi) d\phi d\theta}{2\pi a^n \int_0^{\pi} (P_n(\cos(\phi)))^2 \sin(\phi) d\phi},$$

and

$$A_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^{\pi} F(\theta, \phi) \cos(m\theta) P_n^m(\cos(\phi)) \sin(\phi) d\phi d\theta}{\pi a^n \int_0^{\pi} (P_n^m(\cos(\phi)))^2 \sin(\phi) d\phi},$$

and

$$B_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^{\pi} F(\theta, \phi) \sin(m\theta) P_n^m(\cos(\phi)) \sin(\phi) d\phi d\theta}{\pi a^n \int_0^{\pi} (P_n^m(\cos(\phi)))^2 \sin(\phi) d\phi}.$$

